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Ye.N. Polyakhova

Collection of Problems on the Dynamics of a Point in a Central Force Field

Translation of Sbornik zadach po dinamike tochki v pole tsentral'nykh sil, Leningrad, Leningrad University Press, 1974, pp. 1-145.

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16. Abstract

The collection is a detailed selection of problems on the dynamics of the motion of a material point acted on by a central gravitational force, in particular, the dynamics of space flight. As an exception, the book presents several problems on the motion of a point acted on central nongravitational forces. The book is written mainly for correspondence students. Topics covered include Kepler's laws, the integral of areas, Binet's formulas for central forces, the energy balance and velocity along a space trajectory, time of motion along a space trajectory, conditions for the existence of elliptical trajectories, transfer fromoorbit to orbit, sphere of action, third escape velocity problems, two-body problem, and the generalized third law of Kepler, along with miscellaneous problems.

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AUTHOR'S ABSTRACT

Sbornik zadach po dinamike tochki v pole tsentral'nykh sil, [Collection of Problems on the Dynamics of a Point in a Central Force Field], Ye. N. Polyakhova, Leningrad, Leningrad University Press, 1974, pp. 1-145.

The collection is a detailed selection of problems on the dynamics of the motion of a material point acted on by a central gravitational force of attraction, in particular, the dynamics of space flight. As an exception, the book presents several problems on the motion of a point acted on by central nongravitational forces.

The collection is intended mainly for correspondence students, however it can also be used as a text in the course on theoretical mechanics for students in day and evening departments. It may also prove useful to instructors providing practical exercises in the course on theoretical mechanics. 37 illustrations, 6 tables, 9 references.

FOREWORD

This collection is a textbook for the course of theoretical mechanics ("Point Dynamics" section). It is intended mainly for students in correspondence departments of Leningrad State University and other higher educational institutions. It may also be used in part by students in the day and evening departments. Moreover, the problem book may prove useful for beginning instructors in providing practical exercises in the course on theoretical mechanics, particularly, when they prepare modifications of test problems.

Most of the collection is a detailed sampling of problems on the dynamics of a material point acted on by gravitational force, in particular, problems on the elementary dynamics of space flight. Several problems on the motion of a point acted on by central non-gravitational forces are presented. Altogether, the collection includes about 200 problems of varying degrees of difficulties, with solutions.

The basis for Sbornik zadach was material from lectures and practical exercises in the course on theoretical mechanics given by the author for a number of years in the mathematics and mechanics division of Leningrad State University. The conditions of the problems were set up by the author or else were taken from various domestic and foreign texts, indicated in the bibliography. Solutions to all problems, as well as related computations, have been provided anew by the author or have been carefully verified.

It should be noted that the collection presents solutions of most of the problems proposed in the new chapter "Dynamics of Space Flight" from Sbornik zadach po teoreticheskoy mekhanike [Collection of Problems on Theoretical Mechanics] by I. V. Meshcherskiy.

The proposed problems, as related to their subject matter, are grouped into 11 sections. At the beginning of each section (not including the last) a brief theoretical background and essential formulas on the dynamics of a point are presented. The section "Miscellaneous Problems" groups problems in whose solution information from various parts of the collection is required.

A distinguishing feature of this Sbornik zadach is the exposition of a number of theoretical questions on the dynamics of a point in a central force field in the form of problems, so that quite often references to the numbers of formulas are derived directly in the solution are encountered in the text. The author

hopes that the presentation of several elements of theory in the form of problems will promote an easier grasp of the material when presented in the correspondence teaching form.

The author is obliged to thank the head of the celestial mechanics faculty of the mathematics and mechanics department of Leningrad State University, Professor K. V. Kholshevnikov and the docent of the theoretical mechanics department of this same faculty, S. A. Zegzhda, for attentively reading the manuscript, for valuable counsel and comments that did much to promote improvement in the book. The author will also be appreciative of all who wish to report errors found or to express their critical comments and suggestions.

TABLE OF CONTENTS

CHAPTER ONE

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CENTRAL FORCES. FORCE OF GRAVITY AND ITS DYNAMIC CHARACTERISTICS

In mechanics a force $\bar{F}$ whose line of action extends through a point called the force center (center of attraction) is called a central force. The central force can be expressed by the formula

$$\bar{F} = F \cdot \bar{r}^0, \tag{1.1}$$

where $\bar{r}^0$ is the unit vector of the radial direction connecting the force center with the point of its application. In attraction the magnitude of force F is negative, and in repulsion -- -- positive.

Central forces are subdivided into forces of attraction (directed toward the center) and forces of repulsion (acting from the center). Among the central forces most widespread in nature we should note first of all mutual attraction, that is, the gravitational force of attraction. Its law of variation, formulated by Newton, is known as the law of universal gravity.

The law of universal gravity states that the masses m and M mutually attract each other with a force directly proportional to the product of these masses and inversely proportional to the square of the distance r between them:

$$|F| = \frac{fmM}{r^2}. \tag{1.2}$$

The coefficient of proportionality $f = 6.673 \cdot 10^{-8}$ $cm^3/g \cdot sec^2$ is called the universal gravitational constant.

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* Numbers in the margin indicate pagination in the foreign text.

From Newton's second law it follows that masses and accelerations in a gravity field are associated by the relation

$$\frac{m}{M} = \frac{w_M}{w_m}. \tag{1.3}$$

According to Eq. (1.3) two problems in the dynamics of a point in a central gravity force field are distinguished:

1) the motion of a point with small mass m in the gravity field of a point with large mass M ($m \ll M$), where the ratio of masses is such that acceleration w_M can be neglected and therefore we can assume mass M to be fixed (the limited problem of two bodies, or the problem of the motion of a "nonattracting" point in the gravity field of a "attracting" point); and

2) the motion of a point with mass m in the gravity field of a point with mass M ($m < M$), when acceleration w_M cannot be neglected (the problem of two bodies, or the problem of the motion of one "attracting" point in the gravity field of another). /6

Problems of the first type include all problems on the motion of artificial satellites and spacecraft in central gravity fields of planets for the Sun, that is, on the dynamics of space flight. Their solution then makes up the bulk of the collection. Problems of the second type are examined only as exceptions.

If not specifically stipulated, we will not differentiate from the dynamic point of view the concept of the gravitationally attracted material point and the gravitationally attracting body: in the class of these problems we will assume bodies whose dimensions cannot be neglected to be homogeneous bodies, and the attraction of a homogeneous sphere, as we know, is equal to the attraction of a material point coincident with the center of the sphere at which the entire mass of this sphere is concentrated.

Problem 1.1. Set up an equation of the limited two-body problem, that is, equations of motion of a point with mass m in a field of central attractive force with fixed point mass M ($m \ll M$).

Solution. Let us place at the point of mass M the origin of an absolute inertial coordinate system x, y, z. The position of the point with mass m relative to this origin we will characterize as the radius-vector $\bar{r} = r\bar{r}^0$. The central attractive force acts on mass m. According to Eqs. (1.1) and (1.2) this force is of the form:

$$\bar{F} = F\bar{r}^{0}, \quad F = -\frac{fmM}{r^2}, \tag{1.4}$$

and the equations of motion of a point acted on by this force are

$$m\frac{d^2\bar{r}}{dt^2} = -\frac{fmM}{r^2}\bar{r}^{0}, \quad \text{or} \quad m\frac{d^2\bar{r}}{dt^2} = -\frac{fmM}{r^3}\bar{r}.$$

By dividing both parts of the equality by m and denoting $fM = \mu$, we get finally

$$\frac{d^2\bar{r}}{dt^2} + \mu\frac{\bar{r}}{r^3} = 0. \tag{1.5}$$

Eq. (1.5) is called the vector equation of motion for a limited two-body problem. From this equation it follows that the motion (acceleration of a point depends only on the gravitationally attracted point mass M concentrated at the origin of coordinates.[1]
The coefficient $\mu = fM$, called the gravitational parameter of the /7
central body, characterizes the intensity of the gravitational field induced by mass M.

The vector equation (1.5) is equivalent to three equations expressed in coordinate form:

$$\frac{d^2x}{dt^2} + \mu\frac{x}{r^3} = 0, \quad \frac{d^2y}{dt^2} + \mu\frac{y}{r^3} = 0, \quad \frac{d^2z}{dt^2} + \mu\frac{z}{r^3} = 0. \tag{1.5'}$$

Problem 1.2. Derive the formula of action of a central attractive force of fixed mass M as a point with mass m moves in the field of this force.

Solution. To derive the formula of the elementary work done by the attractive force, let us use the expression for force (1.4):

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[1] This mass is also called the central mass, or the mass of the central body (if the body is a homogeneous sphere), since the line of action of the force continuously passes through this point mass, also called the force center. The term "central mass" by no means signifies that this mass is at the center of the trajectory of motion, although in a particular case this is possible.

$$\delta A = \bar{F}\cdot d\bar{r} = -\frac{fmM}{r^2}(\bar{r}^0\cdot d\bar{r}) = -\frac{fmM}{r^3}(\bar{r}\cdot d\bar{r}) = -\frac{fmM}{r^2}r\,dr. \quad (1.6)$$

By integrating (1.6), we get expressions for the work done by the attractive force as a point mass m at the surface of the imaginary sphere with radius r_1 is moved to a new position on the surface of a sphere with radius r_2:

$$A = \int_{r_1}^{r_2}\left(-\frac{fmM}{r^2}\right)dr = \frac{fmM}{r^2}\Bigg|_{r_1}^{r_2} = fmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \quad (1.6')$$

If a point with mass m approaches mass $M(r_2 < r_1)$, the work done by the attractive force is positive. If point m separates from this mass $(r_2 > r_1)$, the work is negative.

Problem 1.3. Show that the attractive force is a potential force and determine the form of the potential function.

Solution. The function of coordinates U whose differential is equal to the elementary work done is called a potential function, or a force function, while the force and force field for which this function exists are potential force and force fields.

Based on Eq. (1.6) we conclude that $dU = \delta A = \bar{F}\cdot d\bar{r} = -\frac{fmM}{r^2}dr$. Thus, the potential function exists and is of the form

$$U = \int\left(-\frac{fmM}{r^2}\right)dr = \frac{fmM}{r} + C.$$

We can determine the constant C from the relation

$$C = \lim_{r\to\infty} U = U_\infty, \quad (1.7)$$

that is, C is the value of the potential function at infinity, and it can be set equal to zero: $C = U_\infty = 0$. By thus fixing /8
the arbitrary constant, we get the expression:

$$U = \frac{fmM}{r} + U_\infty = \frac{fmM}{r}, \quad (1.7')$$

which we will call the potential of the attractive force (the potential function with fixed arbitrary constant is usually called the potential).

The set of services of the potential level for different r values is a set of concentric spheres whose common center coincides with the center of attraction where mass M is concentrated. Equalities (1.7) and (1.7') enable us to write

$$\frac{fmM}{r} = U - U_{\infty} = A_{\infty,r}, \tag{1.7"}$$

so that the gravitational potential at a given surface of a level, that is, at the surface of a sphere with given radius is equal to the magnitude of the work done by the attractive force (potential difference) as point mass m is moved from infinity to this surface of the level. The attractive force can be expressed here in terms of the potential:

$$F = \partial U/\partial r = -fmM/r^2.$$

Problem 1.4. Show that the gravitational potential of a homogeneous sphere with mass M is equal to the potential of a point mass m equal to the mass of the sphere concentrated at its center.

Solution. Knowing the expression for the gravitation potential of point mass (1.7'), we can compute the potential for point mass m attracted by a homogeneous sphere with mass M and radius R. Let us subdivide the elementary volume of the sphere $d\tau$ and calculate it in a system of spherical coordinates ρ, θ, and ϕ (Fig. 1) bound with a fixed direction to point mass m at distance r from the center of the sphere. We can easily see that the elementary volume $d\tau = d\rho \cdot \rho d\theta \cdot \rho \sin\theta d\phi$, therefore the potential dU of this volume for point m at distance h from element $d\tau$ will be $dU = \frac{fmdM}{h} = \frac{fm\nu d\tau}{h}$, where $h = \sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}$; ν is the density of the sphere. By integrating in ρ from 0 to R, in θ from 0 to π, and in ϕ from 0 to 2π, for the entire sphere we get

$$U = fm\nu \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\rho^2 d\rho \sin\theta \, d\theta \, d\varphi}{\sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}} =$$ /9

$$= 2\pi fm\nu \int_0^R \int_0^\pi \frac{\rho^2 d\rho \sin\theta \, d\theta}{\sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}}.$$

We set $u = u(\theta) = r^2 + \rho^2 - 2r\rho\cos\theta$, so that $du = 2r\rho\sin\theta\, d\theta$. Therefore

$$\int_0^\pi \frac{\rho\sin\theta\, d\theta}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}} = \frac{1}{r}\int_{u_1}^{u_2}\frac{du}{2\sqrt{u}} = \left[\frac{1}{r}\left(r^2+\rho^2-2r\rho\cos\theta\right)^{\frac{1}{2}}\right]_0^\pi =$$

$$= \frac{1}{r}\left[(r+\rho) - |r-\rho|\right] = \frac{2\rho}{r}.$$

So

$$U = \frac{4\pi f m \nu}{r}\int_0^R \rho^2 d\rho = \left(\frac{4}{3}\pi R^3 \nu\right)\frac{fm}{r}.$$

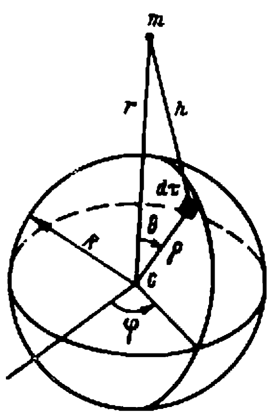

Fig. 1

But the mass of the homogeneous sphere $M = \frac{4}{3}\pi R^3 \nu$, so the gravitational potential of a homogeneous sphere with mass M for point m at distance r from the center is

$$U = \frac{fmM}{r}, \qquad (1.8)$$

that is, it is equal to the potential of point mass M concentrated at the center of the sphere. Here the force of gravitational attraction of the sphere is also equal to the central force of attraction of the point mass:

$$F = F_r = \operatorname{grad} U = \frac{\partial U}{\partial r} = -\frac{fmM}{r^2}.$$

We note that the result is valid not only for a homogeneous sphere, /10
but also for a sphere with spherical distribution of densities, that is, when the densities are equal at points equidistant from the center. In this case $M = 4\pi\int_0^R \nu(\rho)\rho^2 d\rho$.

Based on the foregoing we can conclude that two spheres (homogeneous or with spherical distribution of densities) are attracted as corresponding material points. This important property of the potential of the sphere enables us within the framework of the problems considered in this collection to assume the gravitation of planets and the Sun to be gravitation of point masses, considering that the distribution of the densities of these bodies can be assumed spherical, with adequate accuracy.

Problem 1.5. An artificial Earth satellite with mass m is moving uniformly in a circular radius at given altitude H acted on by the Earth's gravitational force. Write the law of action of the force as a function of angular and linear velocities of motion. Determine how many revolutions per day are made by the "zero", that is, fictive, satellite around the Earth, moving along the surface of its spherical Earth in the circular orbit.

Solution. In uniform motion along the circle, the acceleration w of a point consists only of the radial component $w = w_r = -\omega^2(R_{\oplus} + H)$. Here $R_{\oplus}$ is the radius of the Earth, and ω is the angular velocity of the AES. Then in accordance with the equations of motion we can write

$$F = F_r = -m\omega^2(R_{\oplus} + H), \tag{1.9}$$

or

$$F = F_r = -\frac{mv^2}{R_{\oplus} + H}, \tag{1.9'}$$

where the angular velocity ω and the linear velocity v are associated by the relation $v = (R_{\oplus} + H)\omega$. Eqs. (1.9) and (1.9') characterize the desired law of action of the force.

We know that the "zero" satellite ($r = R_{\oplus}$) will move with linear velocity v = 7.9 km/sec (first escape velocity). Let us find its angular velocity:

$$\omega = \frac{v}{R_{\oplus}} = \frac{7.9\,\text{km/sec}}{6370\,\text{km}} \cdot \frac{86400}{6.28} = \frac{7.9 \cdot 8.64 \cdot 10^4}{6.37 \cdot 10^3 \cdot 6.28} \approx 17.1 \text{ revolutions.}$$

Thus, actual Earth satellites cannot make more than 17.1 revolutions per day.

Problem 1.6. An artificial Earth satellite with mass m moves uniformly in a circular orbit at given altitude H acted on by the /11
Earth's gravitational force. Knowing that the mutual attraction of Earth and AES [artificial Earth satellite] obeys Newton's law, express the force of attraction in terms of the acceleration due to gravity at the Earth's surface and at altitude H.

Solution. For this problem Newton's law is of the form $|F| = fmM_{\oplus}/(R_{\oplus} + H)^2$. We know that at the Earth's surface (H = 0)

the attractive force is numerically equal to the weight of the body: $|F| = D = mg$ (we will assume the Earth to be fixed, and thus we will neglect the inertial forces). Hence $\frac{fmM_♁}{R_♁^2} = mg$, or

$$fM_♁ = gR_♁^2. \tag{1.10}$$

The attractive force F can be expressed here in terms of the acceleration due to gravity g:

$$|F| = \frac{mgR_♁^2}{(R_♁+H)^2}, \quad \bar{F} = -\frac{mgR_♁^2}{r^2}\bar{r}^0. \tag{1.11}$$

Note also that from Eq. (1.10) there derives a formula for computing the acceleration due to gravity at the Earth's surface:

$$g = \frac{fM_♁}{R_♁^2}. \tag{1.11'}$$

For arbitrary altitude H, $|F| = \frac{fmM_♁}{(R_♁+H)^2} = \frac{mgR_♁^2}{(R_♁+H)^2} = mg_H$, where g_H is the acceleration due to gravity at altitude H. Finally, we get

$$fM_♁ = gR_♁^2 = g_H(R_♁+H)^2, \quad g_H = g\frac{R_♁^2}{(R_♁+H)^2}. \tag{1.11"}$$

Problem 1.7. To which altitude must a circular-orbit Earth satellite moving in the plane of the equator be inserted into orbit in order to be continuously over the same point on the Earth's surface ($R_♁$ = 6370 km).

Solution. To fulfill this condition for stationary status of the "satellite track" point it is necessary that the angular velocity of the AES in orbit be equal to the angular velocity of the Earth's rotation $\omega_♁$, which can be readily calculated by the formula $\omega_♁ = \frac{2\pi}{P_♁} = \frac{2\pi}{24 \cdot 3600} = 7 \cdot 10^{-5}$ sec^{-1} ($P_♁$ is the period of the Earth's rotation about its axis). This AES is called a diurnal (24-hour) stationary [geostationary] satellite.

To solve the problem, let us use Eq. (1.9), which for our case must be rewritten as $|F| = m\omega_♁^2(R_♁+H)$, and Eq. (1.11), enabling us to express the force in terms of the acceleration

due to gravity. By equating the force expressions, we find the formula for the altitude $H = \sqrt[3]{\frac{gR_\oplus^2}{\omega_\oplus^2}} - R_\oplus$. By substituting in it $g = 9.81$ m/sec^2, $R_\oplus = 6370$ km, we get $H \approx 35{,}800$ km. /12

Problem 1.8. Determine the velocity required for rectilinear vertical climb of a missile with mass m from the Earth's surface to altitude H in the Earth's gravitational field.

Solution. In the ascent of a missile from the Earth's surface to an assigned altitude H there is a change in the kinetic energy of the missile equal to the work done by the Earth's gravitational force (see problem 1.2). In this case we have

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = fmM_\oplus\left(\frac{1}{R_\oplus+H} - \frac{1}{R_\oplus}\right) = -\frac{fmM_\oplus H}{R_\oplus(R_\oplus+H)}.$$

At altitude H the final velocity is equal to zero (the missile comes to a halt), and the initial velocity $v_0 = \sqrt{\frac{2fM_\oplus H}{R_\oplus(R_\oplus+H)}}$, or after replacing $fM_\oplus$ by $gR_\oplus^2$ according to Eq. (1.10), we have

$$v_0 = \sqrt{\frac{2gR_\oplus H}{R_\oplus+H}}. \qquad (1.12)$$

Let us examine the two limiting cases:

1) for $H \ll R_\oplus$ (low altitudes) $v_0 = \sqrt{2gH/\left(1+\frac{H}{R_\oplus}\right)} \approx \sqrt{2gH}$ (Galileo's formula), and

2) for $H \to \infty$ (separation from the Earth's gravitational field), $\lim\limits_{H\to\infty} v_0 = \lim \sqrt{2gR_\oplus/\left(1+\frac{R_\oplus}{H}\right)} = \sqrt{2gR_\oplus} = 11.19$ km/sec (second escape velocity).

Problem 1.9. Determine the velocity of fall of a point with mass m to the Earth's surface if this point is dropped at altitude H without initial velocity.

Solution. The problem is the inverse problem 1.8. In this case there is a change in the kinetic energy of the missile: $\frac{mv^2}{2} - \frac{mv_0^2}{2} = fmM_\oplus\left(\frac{1}{R_\oplus} - \frac{1}{R_\oplus+H}\right) = \frac{fmM_\oplus H}{R_\oplus(R_\oplus+H)}$. Then $v_0 = 0$, but the velocity

of landing is

$$v = \sqrt{\frac{2fM_{♁}H}{R_{♁}(R_{♁}+H)}} = \sqrt{\frac{2\,gR_{♁}H}{R_{♁}+H}}\,. \tag{1.12'}$$

We can see that the velocity of landing found from Eq. (1.12') is the same as that calculated by the formula of the launch velocity for a climb to altitude H (1.12). For $H \ll R_{♁}$, we again have $v \approx \sqrt{2gH}$, and for $H \to \infty$ (arrival from "infinity") $v \approx \sqrt{2gR_{♁}} = 11.19$ km/sec. Hence, in particular, it follows that meteorites falling on the Earth "from infinity" in a parabolic trajectory can have this velocity.

Problem 1.10. Determine the velocity v_0 that must be imparted along a vertical directed upwards to a body at the surface of the /13
Earth in order for the body to ascend to an altitude equal to the Earth's radius ($H = R_{♁} = 6370$ km). Here we will assume only the Earth's gravitational force. The acceleration due to gravity at the Earth's surface $g = 981\ \mathrm{cm/sec^2} = 9.81 \cdot 10^{-3}\ \mathrm{km/sec^2}$.

Solution. Substituting into Eq. (1.12) the value $H = R_{♁}$, let us find $v_0 = \sqrt{2gR_{♁}^2/2R_{♁}} = \sqrt{gR_{♁}} = \sqrt{9.81 \cdot 10^{-3} \cdot 6370} \approx 7.9$ km/sec.

CHAPTER TWO /13

KEPLER'S LAWS

Kepler's laws on planetary motion are usually formulated as follows:

1. Each planet moves in an ellipse, at one focus of which is the Sun.

2. The area of the sector described by the planet's radius-vector changes in proportion to time.

3. Squares of the periods of revolution of the planets relate as cubes of the semi-major axes of their orbits.

Problem 2.1. Show that the coordinates of a planet moving around the Sun according to Kepler's laws can be expressed as a function of time.

Solution. Let us turn to Kepler's first law and consider an ellipse (Fig. 2) described by point M (planet) whose center C we will take as the origin of coordinates. We will use as the Cx axis the direction of the semi-major axis $C\pi = \alpha$ extending through focus S at which the Sun is, and as the Cy axis -- the direction of the semi-minor axis CB = b. Let us replace the equation of an ellipse in this coordinate system

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2.1)$$

by parametric equations, selecting the parameter as follows. From point M (x, y) representing the position of a planet, let us drop /14 a perpendicular MN to the X-axis. Extending this perpendicular upwards until it intersects a circle whose diameter is the major axis $\alpha\pi$, we get point M'. Specifying each of the points M and M' uniquely defines another point. But the position of point M' can be characterized by angle E for the center of an ellipse measured from the semi-major axis $C\pi$ along the line CM' in the direction

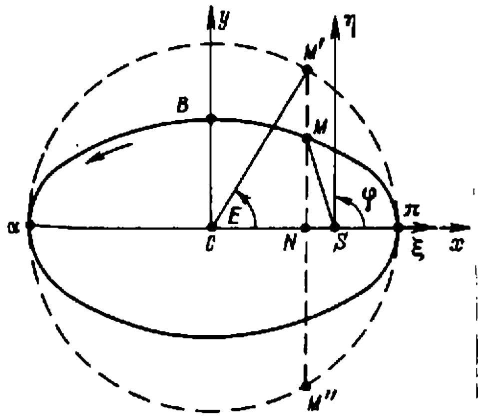

Fig. 2

of planetary motion. This angle is called the eccentric anomaly of the planet. Eq. (2.1) is equivalent to the parametric equations of an ellipse $x = \alpha \cos E$, and $y = b \sin E$, the first of which is obvious from geometrical considerations ($x = CN$), and we obtain the second after substituting the first into the canonical equation (2.1). The ratio $e = CS:C\pi < 1$ defining the shape of the ellipse is its eccentricity. Obviously, $CS = \alpha e$, $CB = b = \alpha\sqrt{1-e^2}$.

Not let us introduce a system of rectangular orbital coordinates $S\xi\eta$, whose axes are parallel to the system axes Cxy, and whose origin is at focus S. Then

$$\xi = CN - CS = \alpha\cos E - \alpha e, \quad \eta = NM = \alpha\sqrt{1-e^2}\sin E. \tag{2.2}$$

Let us denote by r and ϕ the polar orbital coordinates corresponding to ξ and η. The angle ϕ measured from the radius vector $S\pi$ oriented at the perihelion π is called the true anomaly of the planet. Since

$$\xi = r\cos\varphi, \quad \eta = r\sin\varphi, \tag{2.3}$$

then, by comparing equalities (2.2) and (2.3) we get the formulas

$$r\sin\varphi = \alpha\sqrt{1-e^2}\sin E, \quad r\cos\varphi = \alpha(\cos E - e), \tag{2.4}$$

which serve in calculating the polar orbital coordinates r and ϕ for a given E.

From Eqs. (2.4) we can easily determine that

$$r = \alpha(1 - e\cos E). \tag{2.5}$$

Subtracting from (2.5) the second of the equalities (2.4) and then adding them, we find $r(1-\cos\varphi) = \alpha(1+e)(1-\cos E)$, $r(1+\cos\varphi) = \alpha(1-e)(1+\cos E)$, or

$$\sqrt{r}\sin\frac{\varphi}{2} = \sqrt{\alpha(1+e)}\sin\frac{E}{2}, \quad \sqrt{r}\cos\frac{\varphi}{2} = \sqrt{\alpha(1-e)}\cos\frac{E}{2}. \tag{2.6}$$

where in extracting the root the sign is determined uniquely, since the angles $\phi/2$ and E/2 are always in the same quadrant (E = 180° corresponds to ϕ = 180°). From (2.6) there also follows /15

$$\operatorname{tg}\frac{\varphi}{2}=\sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}. \qquad (2.7)$$

Thus, to calculate r and ϕ for a given E we can use, instead of (2.4), Eqs. (2.6) or Eqs. (2.5) and (2.7).

Note that by canceling out E from Eqs. (2.4) and (2.5), we get the equation of an ellipse in the polar coordinates:

$$r=\frac{p}{1+e\cos\varphi}, \qquad (2.8)$$

where $p=a(1-e^2)=b\sqrt{1-e^2}$ is the focal parameter of an ellipse, that is, the ordinate of a point for which ϕ = 90°. Eq. (2.8) is also the general equation of conic sections (when e = 1 it is a parabola, and when e > 1 it is a branch of a hyperbola concave with respect to focus S).

It remains to show that angle E can be expressed as a function of time. To do this we turn to Kepler's second law.

If we let t_π and t refer to the moments of time at which the planet is at the perihelion and at arbitrary point M, and if we let T represent the planetary period of revolution, based on the second law we have

$$\frac{Q}{\pi a b}=\frac{t-t_\pi}{T}, \qquad (2.9)$$

where πab is the area of the ellipse, Q is the area of the focal sector SπM equal to the difference between the areas of the curvilinear trapezium NMπ, and a triangle NMS, that is,

$$Q \text{ is area } NM\pi \text{ -- area } \Delta NMS. \qquad (2.10)$$

To calculate the area of the trapezium NMπ let us use the ratio of the areas of the two curvilinear (elliptical and circular) trapezia:

$$\frac{\text{area } NM\pi}{\text{area } NM'\pi}=\frac{b}{\alpha}. \qquad (2.11)$$

The area of trapezium NM'π in turn is half the area of the circular segment M'πM" and can be determined from the familiar formula of the circular segment with aperture angle 2E: area NM'π = 1/2·
·$\alpha^2/2$ (2E-sin 2E) = 1/2 α^2 (E-sin E cos E), whence by means of Eq. (2.11) we get area NMπ = 1/2 αb (E-sin E cos E). The area ΔNMS appearing in Eq. (2.10) can be determined by means of Eqs. (2.3) and (2.4): area $\Delta NMS = \frac{1}{2}\eta_M(-\xi_M) = -\frac{1}{2}r^2 \sin\varphi\cos\varphi = -\frac{1}{2}\alpha b \sin E(\cos E - e)$.

The "minus" sign in front of ξ_M corresponds to the "minus" sign /16
in Eq. (2.10), from which we finally have Q = 1/2 αb (E - e sin E), owing to which Eq. (2.9) can be written as

$$E - e \sin E = M, \tag{2.12}$$

where $M = n(t - t_\pi)$, $n = \frac{2\pi}{T}$.

The quantity n, the mean rate of change of angle E, that is, the mean angular velocity of the planet, is called the mean motion of the planet, M is the mean anomaly, and Eq. (2.12) is Kepler's equation (see also Section [Chapter] Six).

Kepler's equation enables us to completely solve the problem of determining the angle E as a function of time. Thus, based on the results obtained above it can be stated that the coordinates of a planet moving around the Sun according to Kepler's laws can be expressed as a function of time.

Problem 2.2. Using the formula r = α(1 = e cos E), derive the formulas of the aphelion r_α and perihelion r_π distances of a planet from the Sun and the orbital eccentricity (Fig. 2).

Solution. For the perihelion π and the aphelion α, angle E is 0 and 180°, respectively, so that

$$r_\pi = \alpha(1-e),\ r_\alpha = \alpha(1+e),\ \alpha = \frac{1}{2}(r_\pi + r_\alpha). \tag{2.13}$$

Hence the eccentricity is

$$e = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi}, \tag{2.13'}$$

and the ratio of distances is

$$\frac{r_\pi}{r_\alpha} = \frac{1-e}{1+e}. \tag{2.13''}$$

We can easily see that the mean (arithmetic mean) distance of a planet from the Sun is equal to the semi-major axis:

$$r_{av} = \frac{1}{2}(r_\pi + r_\alpha) = \alpha.$$

Problem 2.3. The semi-major axis of the Earth's orbit as it moves around the Sun is $149.6 \cdot 10^6$ km, and the orbital eccentricity of the Earth $e = 0.01678$. Calculate the largest r_α and the smallest r_π distances from Earth to Sun.

Solution. From Eqs. (2.13) we have $r_\pi = 149.6 \cdot 10^6\,(1-0.01678) \approx$ $\approx 147.1 \cdot 10^6$ km, $r_\alpha = 149.6 \cdot 10^6\,(1+0.01678) = 152.1 \cdot 10^6$ km.

Problem 2.4. A satellite moves around a planet with radius R in an elliptical orbit, whose eccentricity is e. Find the semi-major axis of the orbit if the ratio of the pericenter altitude to the apocenter altitude $H_\pi/H_\alpha = \gamma < 1$.

Solution. For an elliptical orbit we can write $2\alpha = r_\pi +$ /17
$+ r_\alpha = H_\pi + H_\alpha + 2R = H_\alpha\,(1 + H_\pi/H_\alpha) + 2R = r_\alpha - R)(1 + \gamma) + 2R$, but $r_\alpha = \alpha(1 + e)$, therefore $2\alpha - 2R = \alpha(1 + e)(1 + \gamma) - R(1 + \gamma)$, $\alpha[1 - \gamma - e\,(1 + \gamma)] = R$, whence we have $\alpha = R\,(1 - \gamma)/1 - \gamma - e\,(1 + \gamma)$.

Problem 2.5. Determine the mean radius-vector in time $[r]_t$ if α is the semi-major axis of the orbit and e is its eccentricity. Consider the values [r] for averaging with respect to the other variables of elliptical motion.

Solution. By a time-averaged radius-vector of a point moving in an elliptical orbit we mean the quantity $[r]_t = \frac{1}{T}\int^{T} r\,dt$,

where T is the period of revolution. Let us use the formulas of elliptical motion obtained in problem 2.1,

$$r = \alpha(1 - e\cos E), \quad E - e\cos E = n(t - T) = M,$$

and let us replace the variable of integration, changing from t to E in the formulas

$$dE - e\cos E\,dE = n\,dt = dM, \quad dt = \frac{1 - e\cos E}{n}\,dE.$$

Let us rewrite the formula of the time averaging in terms of a new variable, by changing the limits of integration for one revolution of the satellite:

$$[r]_t = \frac{1}{T}\cdot\frac{a}{n}\int_0^{2\pi}(1-e\cos E)^2 dE = \frac{1}{T}\cdot\frac{a}{n}\int_0^{2\pi}(1-$$

$$-2e\cos E + e^2\cos^2 E)dE = \frac{1}{T}\cdot\frac{a}{n}\Big(E - 2e\sin E +$$

$$+\frac{1}{2}e^2 E + \frac{e^2}{4}\sin 2E\Big)\Big|_0^{2\pi} = \frac{2\pi}{T}\cdot\frac{a}{n}\left(1+\frac{1}{2}e^2\right).$$

Considering that the period of revolution T = 2π/n, we finally get

$$[r]_t = a\left(1+\frac{1}{2}e^2\right).$$

We can easily show that the same exact result can be obtained by applying the formula for averaging r with respect to mean anomaly M. The latter is an analog of time in formulas of elliptical motion. The period of its variation is 2π. In this case

$$[r]_M = \frac{1}{2\pi}\int_0^{2\pi} r\,dM,$$

where /18

$$r = a(1-e\cos E),\quad dM = (1-e\cos E)dE,$$

so that

$$[r]_M = \frac{a}{2\pi}\int_0^{2\pi}(1-e\cos E)^2 dE = a\left(1+\frac{1}{2}e^2\right).$$

While averaging over the mean anomaly is equivalent to averaging over time, averaging over the eccentric anomaly E gives a different result:

$$[r]_E = \frac{1}{2\pi}\int_0^{2\pi} r\,dE = \frac{a}{2\pi}\int_0^{2\pi}(1-e\cos E)dE = \frac{a}{2\pi}(E - e\sin E)\Big|_0^{2\pi} = a,$$

that is, we again obtain the arithmetic mean $r_{av.} = a = \frac{r_{min} + r_{max}}{2}$ (see problem 2.2): averaging over E leads to a "loss" of ellipticity; ellipticity is manifested when eccentricity is present in the formulas derived above for $[r]_t = [r]_M$.

Let us estimate r_{av} and $[r]_t = [r]_M$ for the Earth's orbit ($e = 0.01678$). $\alpha = r_{av} = 1.4960000 \cdot 10^{13}$ cm is the exact value of the astronomic unit (Earth-Sun distance). Estimate $[r]_t = [r]_M = \alpha\,(1 + \frac{1}{2} e^2) = 1.0001408 = 1.4962106 \cdot 10^{13}$ cm. Within the limits of accuracy needed to solve problems in this collection, we can neglect the difference between these quantities and assume that $[r]_t \approx r_{av} = \alpha = 149.6 \cdot 10^6$ km.

Averaging over the polar angle ϕ (true anomaly) reduces to computing the integral

$$[r]_\varphi = \frac{1}{2\pi} \int_0^{2\pi} r\, d\varphi .$$

This integral can be easily obtained by means of the subtitution

$$\operatorname{tg} \frac{\varphi}{2} = \sqrt{\frac{1+e}{1-e}}\, \operatorname{tg} \frac{E}{2},$$

and when this is differentiated we have

$$\left(1 + \operatorname{tg}^2 \frac{\varphi}{2}\right) d\varphi = \sqrt{\frac{1+e}{1-e}} \cdot \frac{dE}{\cos^2 \frac{E}{2}} .$$

Replacing tg $\frac{\phi}{2}$ by its value from the substitution, we get

$$1 + \operatorname{tg}^2 \frac{\varphi}{2} = 1 + \frac{1+e}{1-e}\, \operatorname{tg}^2 \frac{E}{2} = 1 + \frac{1+e}{1-e} \cdot \frac{\sin^2 \frac{E}{2}}{\cos^2 \frac{E}{2}} ,$$

which enables us to express $d\phi$ in terms of dE:

$$d\varphi = \sqrt{\frac{1+e}{1-e}} \cdot \frac{dE}{\cos^2 \frac{E}{2} + \frac{1+e}{1-e} \sin^2 \frac{E}{2}} .$$

Using the formulas

$$\cos^2 \frac{E}{2} = \frac{1}{2}(1 + \cos E), \quad \sin^2 \frac{E}{2} = \frac{1}{2}(1 - \cos E),$$

let us find

$$d\varphi = \sqrt{\frac{1+e}{1-e}} \cdot \frac{(1-e)\, dE}{1 - e \cos E} = \frac{\sqrt{1-e^2}\, dE}{1 - e \cos E} = \frac{\alpha}{r} \sqrt{1-e^2}\, dE ,$$

so that we finally get /19

$$[r]_\varphi = \frac{a\sqrt{1-e^2}}{2\pi}\int_0^{2\pi} d\varphi = a\sqrt{1-e^2} = b.$$

CHAPTER THREE

INTEGRAL OF AREAS

The theorem of the variation in the kinetic moment $\bar{K}$ (angular momentum) of a point with mass m moving under the effect of a central force $\bar{F}$ is of the form

$$\frac{d\bar{K}}{dt} = \frac{d(\bar{r} \times m\bar{v})}{dt} = \bar{r} \times \bar{F} = 0 .$$

Hence follows the law of conservation of kinetic moment when the point is moving under the influence of central forces, usually called the integral of areas:

$$\bar{r} \times \bar{v} = \bar{c}, \qquad (3.1)$$

where the vector constant $\bar{c}$ is the constant of the areas.

To determine the geometrical meaning of $\bar{c}$, let us introduce the vector $\Delta\bar{\sigma}$, whose modulus is the area ΔOMM' (Fig. 3):

$$\Delta\bar{\sigma} = \frac{1}{2}(\bar{r} \times \Delta\bar{r}).$$

Dividing both parts of the equality by Δt and letting Δt tend to zero, we get the differential formula

$$2\,\frac{d\bar{\sigma}}{dt} = \bar{r} \times \frac{d\bar{r}}{dt} = \bar{r} \times \bar{v} . \qquad (3.2)$$

where $d\bar{\sigma}$ is a vector whose modulus is equal to the elementary area swept by radius vector r in time dt. A comparison of (3.2) with (3.1) enables us to write

$$\bar{c} = 2\frac{d\bar{\sigma}}{dt} . \qquad (3.2')$$

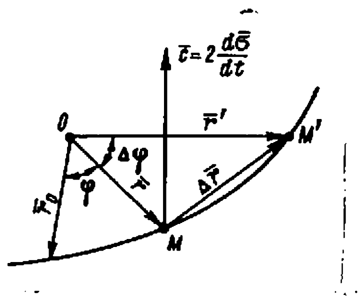

Fig. 3

The quantity $d\bar{\sigma}/dt$ is called the sectorial velocity of a point so that the constant of areas is equal to double the sectorial velocity of a point. The modulus of the sectorial velocity is equal to the velocity of variation in the area σ swept by radius vector $\bar{r}$. Its dimension is length^2/time.

The vector of sectorial velocity is perpendicular to the plane containing the vectors $\bar{r}$ and $\bar{v}$, that is, the areas of the trajectory of a point. From the integral of areas it follows that when a point moves under the influence of central forces the /20
sectorial velocity is constant in magnitude and direction just like the constant of areas. In the particular case, this assertion is valid also for the force of attraction.

Problem 3.1. Show that computing the integral of areas (3.1) is sufficient and necessary for the motion of a point to occur in a single plane passing through the center of attraction.

Solution. Let us place at the attracting center O a rectangular system of axes xyz of arbitrary orientation. Let the components $\bar{C}$ in these axes be denoted by c_x, c_y, and c_z, that is, $\bar{C} = \bar{c}\,\{c_x, c_y, c_z\}$. Then the vector equality (3.1) can be written as three scalar quantities: $y\dot{z} - z\dot{y} = c_x$, $z\dot{x} - x\dot{z} = c_y$, and $x\dot{y} - y\dot{x} = c_z$. Multiplying these equalities by x, y, z, respectively, and adding the results, we arrive at the equation

$$c_x x + c_y y + c_z z = \bar{c}\cdot\bar{r} = 0, \tag{3.3}$$

which is the equation of a plane extending through the origin of coordinates, that is, through the attracting center, and perpendicular to $\bar{C}$. For a central force of attraction these results mean that the motion of a planet occurs in an unchanged plane extending through the center of the Sun. This factor is reflected in Kepler's first law (see Chapter Two).

Problem 3.2. Using the vector formula of sectorial velocity (3.2), find the law of variation of area σ swept by a radius vector, with time.

Solution. From vector equality (3.2) follows the scalar equality $\frac{d\sigma}{dt} = \frac{1}{2}c$, and integrating it gives us $\sigma = \frac{1}{2}ct + \sigma_0$.

This law of linear increase in area σ is written for an arbitrary central force. For the particular case of an attractive force, this result corresponds to Kepler's second law (see Chapter Two).

Problem 3.3. Find the sectorial velocity $d\sigma/dt$ and constant of areas c of the elliptical motion of a point acted on by a central force. Determine the values of the same quantities for the motion of the Earth around the Sun, assuming the Earth's orbit to be circular.

Solution. When a point moves in an ellipse acted on by an arbitrary force its radius-vector sweeps out a total area of an ellipse πab in one period of revolution T, so that /21

$$\frac{d\sigma}{dt} = \frac{\pi a b}{T}, \quad c = \frac{2\pi a b}{T}. \tag{3.4}$$

In particular, this result is valid also for the case when a point moves in an ellipse acted on by a gravitational force of attraction, when the attracting center is at a focus of the ellipse. Taking the Earth's orbit as circular, we have $a = b = 150 \cdot 10^6$ km, T = 365 so that $d\sigma/dt = 1.94 \cdot 10^{14}$ km^2/days $= 0.86 \cdot 10^{-2}$ astronomical unit^2/ /day.

Problem 3.4. Using the integral of areas determine the relation between the planetary velocities at the perihelion and aphelion (Fig. 4).

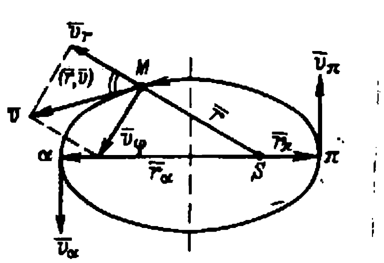

Fig. 4

Solution. The vector equality $\bar{r} \times \bar{v} = \bar{c}$ is satisfied for any point on the orbit, including for the apsides (perihelion π and aphelion α):

$$\bar{r} \times \bar{v} = \bar{r}_\pi \times \bar{v}_\pi = \bar{r}_\alpha \times \bar{v}_\alpha = \bar{c},$$

where the modulus of the constant of areas c can be determined by the formula

$$c = rv \sin(\widehat{\bar{r}, \bar{v}}) > 0, \tag{3.5}$$

where $(\widehat{\bar{r}, \bar{v}})$ is the angle between vectors $\bar{r}$ and $\bar{v}$ at an arbitrary point M (Fig. 4). The quantity c is always positive, so that the

angle $(\bar{r}\hat{,}\bar{v})$ does not exceed 180°. At the apsides the vectors $\bar{r}$ and $\bar{v}$ are orthogonal ($\sin(\bar{r};\bar{v}) = 1$), therefore the following relation is valid

$$\frac{v_\pi}{v_\alpha} = \frac{r_\alpha}{r_\pi}, \qquad (3.6)$$

which means that the velocity of a point at the apsides is inversely proportional to the distance between them and the ellipse focus S (that is, from the Sun). At the perihelion the planetary velocity will be in the greatest, and at the aphelion -- the smallest.

Remark. In generalizing these formulas to the case of any conic section and considering the orthogonality $\bar{r}$ and $\bar{v}$ at the pericenter of this section, we can write: for the ellipse $c = r_{min} \cdot v_{max} = r_{max} \cdot v_{min}$, and for the parabola and hyperbola $c = r_{min} \cdot v_{max}$.

For an arbitrary conic section the following formula is valid:

$$\sin(\bar{r}\hat{,}\bar{v}) = \frac{c}{rv} = \frac{r_{min}\, v_{max}}{rv}.$$

Problem 3.5. If a space rocket at an altitude 230 km over the Earth's surface is given a velocity 10.00 km/sec parallel to the Earth's surface, its orbital apogee will be roughly 370,000 km from the Earth's center (near the orbit of the Moon). What velocity will the rocket have at its apogee? /22

Solution. The computations can be made directly by Eq. (3.6). Assuming the mean radius of the Earth $R_♁ = 6370$ km, let us find

$$v_\alpha = v_\pi \frac{r_\pi}{r_\alpha} = 10.00\,\frac{6370+230}{6370+370{,}000} = \frac{66{,}000}{376{,}370} \approx 0.18 \text{ km/sec}.$$

Problem 3.6. Two meteorites describe the same ellipse, at focus S of which the Sun is situated. The distance between them is so small that the arc M_1M_2 of the ellipse can be assumed to be a segment of a straight line. We know that the distance M_1M_2 is α when its midsection is at the perihelion π (Fig. 5). Assuming that the meteorites will move at equal sectorial velocities, determine the distance M_1M_2 when its midsection will pass through the aphelion. The distances r_π and r_α are known.

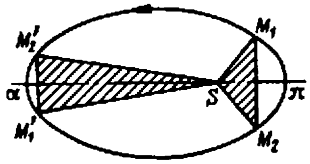

Fig. 5

Solution. Let us denote the symmetric positions of meteorites near the aphelion by M_1' and M_2'. The condition of smallness of distance between the meteorites enabling us to approximate the arc of the ellipse with a chord lets us use, instead of the formula of the area of an elliptical sector, the formula of the area of a triangle. In the time the meteorite M_1 transits the arc M_1M_1', its radius-vector sweeps out the area M_1SM_1'. On the condition that the sectorial velocities are equal to each other, this area is equal to the area M_2SM_2' swept out by the radius vector of meteorite M_2 as the latter transits the arc M_2M_2'. From the equality of these areas there follows the equality of the areas of the triangles M_1SM_2 and $M_1'SM_2'$, that is, $\frac{1}{2}\alpha r_\pi = \frac{1}{2}(M_1'M_2')r_\alpha$, from whence we have $M_1'M_2' = \alpha\frac{r_\pi}{r_\alpha}$.

Since $r_\pi < r_\alpha$, close to the aphelion the meteorites prove to be closer to each other than close to the perihelion (we have in mind the symmetric positions). Here their distance from each other near the aphelion proves to be at a minimum of all possible distances.

Problem 3.7. Describe in polar coordinates the equations of motion of a point acted on by a central field and obtain the first integral of the equations of motion in the scalar form. Show that the resulting integral is an integral of areas. /23

Solution. The equations of motion of a point acted on by a central force $\overline{F} = \overline{F}_r$ in the polar coordinates r and ϕ are two equations in projections in the radial and transversal directions:

$$F_r = m(\ddot{r} - r\dot{\varphi}^2), \quad F_\varphi = \frac{m}{r}\cdot\frac{d}{dt}(r^2\dot{\varphi}) = 0. \tag{3.7}$$

From the second equation at once there follows the first integral

$$r^2\dot{\varphi} = c. \tag{3.8}$$

To show that it is identical with the integral of areas presented above, let us show that the constant c from (3.8) is in fact the constant of areas. Actually, by turning again to Fig. 3, for the

area OMM' swept by $\bar{r}$ in time dt we get area OMM' = $d\sigma = \frac{1}{2} r^2 d\phi$, from whence there follows $r^2\dot{\phi} = 2\frac{d\sigma}{dt} = c$. The identity has been proven.

Problem 3.8. Using the notation for the integral of areas in polar coordinates (3.8), derive a formula for the radial v_r and transversal v_ϕ projections of velocity v of a point as it moves in a conical section (Fig. 4).

Solution. When a point moves in a conical section $r = p/(1 + e\cos\phi)$, this problem consists of determining the quantities

$$v_r = \frac{dr}{dt} = \frac{dr}{d\varphi}\cdot\frac{d\varphi}{dt}, \quad v_\varphi = r\frac{d\varphi}{dt}, \text{ and } \quad v = \sqrt{v_r^2 + v_\varphi^2}.$$

We obtain the derivative $dr/d\varphi = \frac{pe\sin\varphi}{(1+e\cos\varphi)^2}$ from the equation of the section, and the derivative $d\varphi/dt = c/r^2$ from the integral of areas. Finally, we have

$$v_r = \frac{c}{p} e\sin\varphi, \; v_\varphi = \frac{c}{p}(1+e\cos\varphi), \; v = \sqrt{\frac{c^2}{p^2}(1+e^2+2e\cos\varphi)}. \tag{3.9}$$

From (3.9) there follow the formulas of elliptical motion for velocities at the apsides:

$$v_\pi = v_\varphi(\varphi=0) = \frac{c}{p}(1+e), v_\alpha = v_\varphi(\varphi=180°) = \frac{c}{p}(1-e). \tag{3.10}$$

Knowing the ratio of velocities and radius-vectors at the apsides (see problem 3.4), let us find /24

$$\frac{v_\pi}{v_\alpha} = \frac{r_\alpha}{r_\pi} = \frac{1+e}{1-e}. \tag{3.11}$$

Substituting into Eq. (3.10) the expression $p = \alpha(1 - e^2)$ and the constant of areas from (3.5) $c = r_\pi v_\pi = r_\alpha v_\alpha$, we again get the familiar expressions (see problem 2.2) $r_\pi = \alpha(1 - e)$, $r_\alpha = \alpha(1 + e)$, and $e = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi}$.

Problem 3.9. A point M with mass m moves around a fixed center O under the effect of a central force that depends only on distance OM = r. Knowing that the velocity of the point

will change according to the law $v = \alpha/r$, where α is a known constant, find the magnitude and direction of force $\overline{F}$ and the trajectory of this point.

Solution. To determine the force $\overline{F}$ let us use the first of the formulas (3.7) $F = F_r = m(\ddot{r} - r\dot{\phi}^2)$, in which we must substitute the values of the quantities $\ddot{r}$ and $r\dot{\phi}^2$. To determine them, let us use the integral of areas $r^2\dot{\phi} = c$ and the formula for the expansion of velocity in polar coordinates $v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 = (\alpha/r)^2$.

Writing the relations $\dot{\varphi} = \frac{c}{r^2}$, $r\dot{\varphi}^2 = \frac{c^2}{r^3}$, $\dot{r}^2 = \frac{\alpha^2 - c^2}{r^2}$, and $\ddot{r} = \frac{c^2 - \alpha^2}{r^3}$ and substituting them into the formula for force $F = m\left[\frac{c^2-\alpha^2}{r^3} - \frac{c^2}{r^3}\right] = -m\frac{\alpha^2}{r^3}$, let us determine that the force $\overline{F}$ proves to be the attractive force inversely proportional to the cube of the distance to the points. Obviously, this force is not the force of Newtonian attraction.

We can find the trajectory of the point from the relations $\dot{r} = \sqrt{\frac{\alpha^2 - c^2}{r^2}}$, $\frac{r\,dr}{\sqrt{\alpha^2 - c^2}} = dt$, $r^2\dot{\varphi} = c$, and $\frac{r^2}{c}\,d\varphi = dt$, from which after canceling out dt there follows $\frac{r\,dr}{\sqrt{\alpha^2-c^2}} = \frac{r^2}{c}\,d\varphi$. By integrating the left and right parts of the equation, we get

$$\frac{c}{\sqrt{\alpha^2 - c^2}}\ln r = \varphi + C, \quad \text{where} \quad C = \frac{c}{\sqrt{\alpha^2 - c^2}}\ln r_0 .$$

Hence it follows that the equation of the family of logarithmic spirals $r = r_0 e^{\frac{\varphi}{c}\sqrt{\alpha^2 - c^2}}$ is the equation of the set of trajectories.

CHAPTER FOUR /24

BINET'S FORMULAS FOR CENTRAL FORCES

Velocity formula. Suppose a point is moving under the influence of central forces. We know that in the polar coordinates r and ϕ the velocity of the point is expressed by the formula /25

$$v^2 = v_r^2 + v_\varphi^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2.$$

Transforming the expressions for v_r and v_ϕ by means of the integral of areas $r^2 \frac{d\varphi}{dt} = c$, we find

$$v_r = \frac{dr}{dt} = \frac{dr}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{c}{r^2}\cdot\frac{dr}{d\varphi},$$

and

$$v_\varphi = r\frac{d\varphi}{dt} = \frac{c}{r}.$$

Let us introduce the new variable u = 1/r. Then considering that

$\frac{du}{d\varphi} = -\frac{1}{r^2}\cdot\frac{dr}{d\varphi}$, we have $v_r = -c\,\frac{du}{d\varphi}$, and $v_\varphi = cu$.

Finally we get the formula of the square of the velocity of the point moving in the central force field,

$$v^2 = c^2\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right], \qquad (4.1)$$

which is usually called Binet's first formula.

Acceleration formula. Writing out the theorem on the change in kinetic energy of a point moving under the influence of a central force and dividing both parts of the equation by $d\phi$, we get

$$\frac{m}{2}\cdot\frac{d(v^2)}{d\varphi} = F_r\,\frac{dr}{d\varphi}.$$

Let us replace v^2 by its value from Binet's first formula:

$$\frac{mc^2}{2} \cdot \frac{d}{d\varphi}\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] = F_r \frac{dr}{d\varphi}.$$

Using the formula

$$\frac{dr}{d\varphi} = -r^2 \frac{du}{d\varphi} = -\frac{1}{u^2} \cdot \frac{du}{d\varphi},$$

let us write the expression

$$\frac{mc^2}{2}\left[2\frac{du}{d\varphi} \cdot \frac{d^2u}{d\varphi^2} + 2u\frac{du}{d\varphi}\right] = -F_r \frac{1}{u^2} \cdot \frac{du}{d\varphi},$$

and elementary transformation of this formula leads us to

$$w_r = \frac{F_r}{m} = -c^2u^2\left(\frac{d^2u}{d\varphi^2} + u\right). \qquad (4.2)$$

This expression of the acceleration of a material point moving under the influence of a central force is called Binet's second formula. Here the vector of acceleration is defined thusly:

$$\bar{w} = \bar{w}_r = w_r \bar{r}^0 = \frac{F_r}{m}\bar{r}^0, \quad \bar{w}_\varphi = 0. \qquad (4.3)$$

Problem 4.1. Using Binet's formulas, find the law of action of a central force in which a point is moving along a circle with radius r = R. Determine the dependence of force on velocity.

Solution. From Binet's second formula (4.2) there follows the expression of the force $F_r = -mc^2u^3 = -\frac{mc^2}{R^3} = \text{const}$,

that is constant in magnitude. From Binet's first formula (4.1) we find

$$v = cu = \frac{c}{R} = \text{const}, \quad \text{and} \quad c = vR = \text{const}.$$

so that

$$F_r = -\frac{mv^2}{R} = \text{const}.$$ /26

Thus, the motion of a point along a circle originates under the action of a magnitude-constant attractive force $\bar{F}_r$ at velocity

$\bar{v}$, also constant in magnitude. This force is called the centripetal force. The acceleration corresponding to it is also constant in magnitude and coincides completely with the centripetal component.

Problem 4.2. A central force causes a point to move in a logarithmic spiral $r = r_0 e^{\lambda\phi}$ (λ = ctg α) (Fig. 6). Find the law of action of a force by using Binet's formulas.

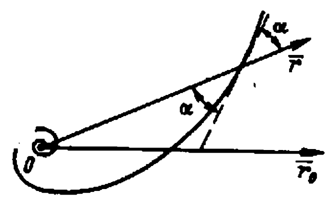

Fig. 6

Solution. Based on the equation of the motion of a point, let us set up auxiliary equations:

$$u = \frac{1}{r_0} e^{-\lambda\varphi},$$

$$\frac{du}{d\varphi} = -\frac{\lambda}{r_0} e^{-\lambda\varphi} = \lambda u, \quad \frac{d^2u}{d\varphi^2} = \frac{\lambda^2}{r_0} e^{-\lambda\varphi} = \lambda^2 u,$$

which must be substituted into Binet's second formula:

$$F_r = -mc^2u^3(\lambda^2+1) = -\frac{mc^2(\text{ctg}^2\alpha+1)}{r^3} < 0,$$

so that the force F_r is an attractive force directed toward the asymptotic point of spiral 0, which is a dynamic force center. This force is proportional to $1/r^3$ and is among the forces of the same type as those examined in problem 3.9. Forces of this type are encountered, for example, in the theory of motion of microparticles.

The velocity of a point defined by Binet's first formula varies inversely proportional to the distance of the point from the center

$$v = c\sqrt{\left(\frac{du}{d\varphi}\right)^2 + u^2} = cu\sqrt{\lambda^2+1} = \frac{c}{r}\sqrt{\text{ctg}^2\alpha+1},$$

so that the velocity increases with approach to the center and vice versa (the direction of velocity can be arbitrary).

Based on the assigned r and v, we can determine the constant of areas

$$c = \frac{rv}{\sqrt{\text{ctg}^2\alpha+1}} = rv\sin\alpha = \text{const.} \tag{4.4}$$

This same relation follows directly from the integral of areas

$$c = |\bar{r} \times \bar{v}| = rv \sin(\bar{r}, \hat{\bar{v}}) = \text{const.}$$

Problem 4.3. Using Binet's formulas, find the exact law of action of a central force directed toward the focus of an ellipse, by using the equation of an ellipse in the polar coordinates $r = p/(1 + e \cos \phi)$, where $p = \alpha(1 - e^2)$. (Direct problem of dynamics for a central force.) /27

Solution. We know that if a point moves according to Kepler's laws, the force causing this motion is a central force directed at the focus of an ellipse. Based on the equation of the trajectory of the point, let us set up auxiliary expressions:

$$u = \frac{1 + e\cos\varphi}{p}, \quad \frac{du}{d\varphi} = -\frac{e}{p}\sin\varphi, \quad \frac{d^2u}{d\varphi^2} = -\frac{e}{p}\cos\varphi,$$

which must be substituted into Binet's second formula to find the law of action of the force

$$F_r = -\frac{mc^2u^2}{p} = -\frac{mc^2}{pr^2},$$

inversely proportional to the square of the distance from the force center. The "minus" sign once again confirms that the force considered is an attractive force. Usually the constant c^2/p is denoted by μ, so that

$$F_r = -\frac{\mu m}{r^2}. \tag{4.5}$$

The coefficient of proportionality μ can be readily determined from the expression $p = \alpha\ (1 - e^2)$ and Eq. (3.4) of the elliptical sectorial velocity in which $b = \alpha\sqrt{1-e^2}$:

$$\mu = \frac{c^2}{p} = \frac{4\pi^2\alpha^2b^2}{pT^2} = \frac{4\pi^2\alpha^3}{T^2} = \text{const.} \tag{4.6}$$

Using Kepler's third law (see Chapter Two), we can easily establish the physical significance of the constant $\mu = c^2/p$. Actually, if α^3/T^2 is constant not only for one body moving in an orbit with given α and T, but is also constant and identical for all bodies moving around the same central body, then it is apparent that the constant characterizes the intensity of the gravitational field produced by this body. The constant μ is called the gravitational

parameter or Gauss' constant. Each body has its own gravitational parameter independent of the gravitational parameters of the other bodies. The relation between the gravitational parameter and the universal gravitational constant f will be determined below, in problem 4.6.

Problem 4.4. Determine the acceleration of a point moving under the law of universal gravity at the moments it transits the pericenter and apocenter. /28

Solution. Based on the law of action of the gravitational force determined in problem 4.4, $F_r = -mc^2/(pr^2)$, so that for acceleration we have

$$w = w_r = -\frac{c^2}{pr^2}, \quad w_\pi = -\frac{c^2}{pr_\pi^2}, \text{ and } \quad w_\alpha = -\frac{c^2}{pr_\alpha^2}.$$

Knowing that

$$p = a(1 - e^2) = a(1+e)(1-e), \quad r_\pi = e(1-e), \text{ and } \quad r_\alpha = a(1+e),$$

we get

$$w_\pi = -\frac{c^2(1+e)^2}{p^3}, \quad w_\alpha = -\frac{c^2(1-e^2)}{p^3}, \quad \frac{w_\pi}{w_\alpha} = \left(\frac{1+e}{1-e}\right)^2.$$

Problem 4.5. Based on the law of action of a central force (4.5) (see problem 4.4), derive the law of universal gravity and find the relation between the universal constant of gravity f with the gravitational parameter μ and the mass of central body M.

Solution. Let us examine any two bodies, for example, Sun and the Earth, with masses $M_\odot$ and $M_\oplus$. Each of these bodies has its own gravitational parameter $\mu_\odot$ and $\mu_\oplus$. The force with which the sun attracts the Earth can be stated using Eq. (4.5):

$$F_\odot = -\frac{\mu_\odot M_\oplus}{r^2}. \tag{4.7}$$

and the force with which the Earth attracts the Sun in accordance with Newton's third law can be written as

$$F_\oplus = -\frac{\mu_\oplus M_\odot}{r^2}. \tag{4.7'}$$

From the law of equality of action and reaction, we get

$$\mu_{\odot} M_{\oplus}/r^2 = \mu_{\oplus} M_{\odot}/r^2, \quad \text{whence}$$

$$\frac{\mu_{\odot}}{M_{\odot}} = \frac{\mu_{\oplus}}{M_{\oplus}} = \ldots = \frac{\mu_{pl}}{M_{pl}} = \text{const}, \tag{4.8}$$

where μ_{pl}/M_{pl} is the ratio of the gravitational parameter of any planet to its mass. Therefore, the ratio of the gravitational parameter of any body to its mass is a constant. It is called the universal constant of gravity, or the constant of universal gravity. Let us denote the constant of gravity by f, and then

$$\mu_{\odot} = fM_{\odot}, \ \mu_{\oplus} = fM_{\oplus}, \ldots, \mu_{pl} = fM_{pl}. \tag{4.9}$$

These formulas establish the relation between the universal gravitational constant with the gravitational parameter and with the mass of the body. Substituting (4.9) into (4.7) and (4.7') /29
enables us to write the law of universal gravitation for the Sun and Earth

$$|F| = \frac{fM_{\odot}M_{\oplus}}{r^2},$$

or for any other two masses m and M

$$|F| = \frac{fMm}{r^2}.$$

The gravitational constants are these: in the SI system $f = 6.673 \cdot 10^{-11}$ $m^3/kg \cdot sec^2$, and in the SGS system $f = 6.673 \cdot 10^{-8}$ $cm^3/g \cdot sec^2$.

Problem 4.6. Knowing that the mass of the Sun $M_{\odot} = 1.97 \cdot 10^{33}$ g, the mass of the Earth $M_{\oplus} = 6.10^{27}$ g, and the mass of the Moon $M_{☾} = 1/81.5$ Earth mass, determine the gravitational parameters of these celestial bodies using the universal gravitational constant $f = 6.67 \cdot 10^{-8}$ $cm^3/g \cdot sec^2$.

Solution. The gravitational parameters of the Sun, Earth, and Moon, calculated by the formula $\mu = fM$, are, respectively, $\mu_{\odot} = 1327 \cdot 10^8$ km^3/sec^2, $\mu_{\oplus} = 398{,}600$ km^3/sec^2, and $\mu_{☾} = 4900$ km^3/sec^2.

Problem 4.7. Knowing the radius of a celestial body R and the acceleration due to gravity at its surface g, determine the gravitational parameter μ of the celestial body and calculate it for the Earth ($R_{\oplus}$ = 6370 km, g = 9.81 m/sec^2).

Solution. Compare the formula $\mu = fM$ with Eq. (1.10) from problem 1.6. As a result we get

$$\mu = gR^2, \tag{4.10}$$

that is, this formula lets us determine the gravitational parameter of any celestial body from the values of g and R. From this formula we can calculate the gravitational parameter of the Earth already calculated by us in problem 4.6:

$$\mu_{\oplus} = fM_{\oplus} = gR_{\oplus}^2 = 398{,}600 \text{ km}^3/\text{sec}^2.$$

Problem 4.8. From $\mu_{\oplus}$ and $g_{\oplus}$ determine the gravitational parameter μ of a celestial body and the acceleration g due to gravity at its surface if we know the ratios of its mass M and radius R to the mass $M_{\oplus}$ and radius $R_{\oplus}$ of the Earth. Compute these quantities for the Moon, Venus, Mars, and Jupiter. The corresponding ratios are given in Table 1.

TABLE 1

Planet	$M/M_{\oplus}$	$R/R_{\oplus}$	Planet	$M/M_{\oplus}$	$R/R_{\oplus}$
Moon	0.123	0.273	Mars	0.107	0.535
Venus	0.814	0.958	Jupiter	314	10.95

Solution. From Eq. $\mu = fM = gR^2$ it follows that the gravi- /30
tational parameters of different bodies relate to each other as the masses of these bodies: $\mu/\mu_{\oplus} = M/M_{\oplus}$, whence $\mu = 398{,}600 \cdot M/M_{\oplus}$ km^3/sec^2, that is, from the mass ratios given in the problem conditions we can calculate the parameters of all these celestial bodies. Also, from this same formula it follows that the accelerations due to gravity at the surfaces of these celestial bodies relate as follows:

$$\frac{g}{g_{\oplus}} = \frac{\mu}{\mu_{\oplus}}\left(\frac{R_{\oplus}}{R}\right)^2 = \frac{M}{M_{\oplus}}\left(\frac{R}{R_{\oplus}}\right)^{-2},$$

whence $g = 9.81\ M/M_{\oplus}\ (R/R_{\oplus})^{-2}$ m/sec^2. By performing the computations, we get the following results (see Table 2).

TABLE 2

Planet	μ, km^3/sec^2	g, m/sec^2
Moon	$4.903 \cdot 10^3$	1.62
Venus	324,460	8.70
Mars	42,650	3.67
Jupiter	$125.16 \cdot 10^6$	25.7

Problem 4.9. Assuming that the central force is a gravitational force $F = -\mu m/r^2$, determine the trajectory of a point under arbitrary initial conditions by using Binet's second formula (Newton's problem, or the inverse problem of dynamics for a gravitational force).

Solution. We know that Binet's second formula (4.2) in conjunction with the law of areas (4.4) gives a system of differential equations from which, by knowing $F = F_r\ (r, \phi, \dot{r}, \dot{\phi}, t)$, we can determine the law of motion of a point acted on by a central force. For the case when F_r does not explicitly depend on time (the gravitational force is specifically in this class of forces), Binet's second formula is a differential equation of the trajectory of the point. From this formula, the second-order inhomogeneous differential equation

$$d^2u/d\varphi^2 + u = -F_r/mc^2u^2 = \mu/c^2;$$

derives for the gravitational force. The solution to this equation must be sought for in the form

$$u = \frac{\mu}{c^2} + A\cos(\varphi + \varepsilon), \tag{4.11}$$

where A and ε are the constants of integration (we can easily

verify that the proposed solution satisfies the equation). Converting in (4.11) from u to $r = 1/u$, we get the formula /31

$$r = \frac{\frac{c^2}{\mu}}{1 + \frac{c^2 A}{\mu} \cos(\varphi + \varepsilon)}, \tag{4.12}$$

which must be compared with the equation of the conic section

$$r = \frac{p}{1 + e\cos\varphi}. \tag{4.12'}$$

From Eq. (4.6) (see problem 4.3) it follows that $p = c^2/\mu$, where μ is the gravitational parameter, so that the numerators of both formulas under comparison are equal to each other. If we set the constant c^2A/μ equal to eccentricity e, and the initial phase of ε equal to zero, the formulas can be assumed to be identical.

The zero-equality of the initial phase means that measuring the polar angle ϕ (at the focus of the conical section), usually called the true anomaly, proceeds from the position of the radius-vector corresponding to the angle $\phi = 0$. Since in Eq. (4.12') $\phi = 0$ corresponds to $r = r_{min}$, then therefore measuring the angles proceeds from the direction toward the pericenter, that is, to the point with polar coordinates ($r_\pi = r_{min}$, $\phi_\pi = 0$).

Thus, the trajectory of a point moving under the influence of the gravitational force, under arbitrary initial conditions, is a conic section, whose shape (ellipse, parabola, or hyperbola) depends on the initial conditions imposed on the motion.

Problem 4.10. Using the equation of a conic section and Binet's first formula, establish the relation between the orbital eccentricity and initial kinematic characteristics of motion r_0 and v_0.

Solution. The equation of a conic section (4.12') allows us to write

$$u = \frac{1 + e\cos\varphi}{p}, \quad e\cos\varphi = up - 1, \quad \frac{du}{d\varphi} = -\frac{e}{p}\sin\varphi. \tag{4.13}$$

Expressing the derivative $du/d\phi$ by means of Binet's first formula, for the initial conditions we can write

$$-\frac{e}{p}\sin\varphi_0 = \left(\frac{du}{d\varphi}\right)_0 = \pm\sqrt{\frac{v_0^2}{c^2} - u_0^2} \;.$$

Replacing p by its value from (4.6), we get

$$e\sin\varphi_0 = \pm\frac{c^2}{\mu}\sqrt{\frac{v_0^2}{c^2} - u_0^2}\,, \qquad (4.14)$$

where the signs correspond to positive and negative values of sin ϕ_0 where $\phi_0 = \varepsilon\ (0,\pi)$ and $\phi_0 = \varepsilon\ (\pi,\ 2\pi)$. /32

From the second equation (4.13) it follows that

$$e\cos\varphi_0 = \frac{u_0 c^2 - \mu}{\mu} \qquad (4.15)$$

(the sign of cos $_0$ is regulated by the numerator).

Taking the squares of (4.14) and (4.15) and adding them, we find the desired relation:

$$e = \sqrt{1 + \frac{c^2}{\mu^2}\left(v_0^2 - 2u_0\mu\right)}\,. \qquad (4.16)$$

As follows from (4.16), eccentricity does not depend explicitly on the polar angle ϕ_0 (the initial value of the true anomaly), however for a given constant of areas c, the relation of ϕ_0 with v_0 and $r_0 = 1/u_0$ is expressed by a formula deriving directly from (4.14) and (4.15):

$$\operatorname{tg}\varphi_0 = \pm\frac{c\sqrt{v_0^2 - u_0^2c^2}}{u_0c^2 - \mu}\,. \qquad (4.17)$$

The law of sign selection here is the same as in (4.14). We note that the constant of areas $c = \sqrt{\mu p}$ of conic sections for an ellipse $c = \sqrt{\mu a(1-e^2)}$ $(e<1)$, for parabola $c = \sqrt{2\mu r_\pi}$ $(e=1)$, and for hyperbola $c = \sqrt{\mu a(e^2-1)}$ $(e>1)$, whence the positive sign of c follows once again.

CHAPTER FIVE /32

ENERGY BALANCE AND VELOCITY ALONG A SPACE TRAJECTORY

In examining problems on the motion of artificial celestial bodies or spacecraft, we must bear in mind that the shape and linear dimensions of trajectories (conic sections) determined by eccentricity e and by the radius-vector of the pericenter r_π depend exclusively on the initial launch conditions. In problem 4.10, the formula relating e with $r_0 = 1/u_0$ and v_0 was obtained, and this formula can be rewritten thusly:

$$e = \sqrt{1 + \frac{c^2}{\mu^2}\left(v_0^2 - \frac{2\mu}{r_0}\right)}. \qquad (5.1)$$

Analysis of the energy significance of (5.1), in particular, the energy significance of the expression appearing in the parentheses, enables us to establish the dependence of the kind of motion on /33 these conditions.

Problem 5.1. A point with mass m is attracted to a fixed center under the law of universal gravity. Write the integral of the energy of the point.

Solution. To write the integral of the energy of a point, let us employ the formula of the potential of the gravitational force (1.7) (see problem 1.3), writing it in the form $U = \mu m/r$, where μ is the gravitational parameter. By selecting the integral of energy in the form $T - U = h$, let us find $mv^2/2 - \mu m/r = h$, where h is the constant of the energy of the point, or the total energy. Dividing both parts of the equation by m, we get

$$v^2 - \frac{2\mu}{r} = \frac{2h}{m} = \tilde{h} = \text{const}, \qquad (5.2)$$

where $\tilde{h}$ denotes the doubled total energy of unit mass.

Customarily, Eq. (5.2) is called the integral of the energy of a point moving in a gravitational force field. It enables us to find the energy significance of $v_0^2 - 2\mu/r_0$ from Eq. (5.1). Actually, since the integral of energy is satisfied for any position of the point in question in orbit, that is,

$$v^2 - \frac{2\mu}{r} = v_0^2 - \frac{2\mu}{r_0} = \tilde{h} = \text{const}, \tag{5.2'}$$

then obviously the difference $v_0^2 - 2\mu/r_0$ is twice the total energy of unit mass (very frequently it is precisely this quantity that is called the energy constant). Now Eq. (5.1) can be rewritten as

$$e = \sqrt{1 + \frac{c^2}{\mu^2}\tilde{h}} \quad . \tag{5.3}$$

Problem 5.2. Based on Eqs. (5.2) and (5.3) derive the formulas for the initial launch velocities needed for insertion of a point into orbit with conic section.

Solution. By specifying in Eq. (5.3) the eccentricity e corresponding to different kinds of conic sections, by means of (5.2') we get the velocities determining each kind of section.

We can readily obtain the formula of local circular velocity as follows. Setting e = 0 in (5.1), let us write

$$v_0^2 - 2\mu/r_0 = -\mu^2/c^2,$$

whence

$$v_0^2 = -\mu(\mu/c^2) + 2\mu/r_0 .$$

To determine μ/c^2, let us use the equations

$$\mu = c^2/p \text{ and } r = r_0 = p/(1 + e\cos\varphi_0) = p,$$

whence

$$\mu/c^2 = 1/p = 1/r_0 \quad \text{and} \quad v_0^2 = \mu/r_0 .$$

In the particular case of the launch of a body from the Earth's surface ($r_0 = R_♁ = 6370$ km), the local circular and /34

local parabolic velocities are called the first and second escape velocities, respectively:[2]

$$v_I = \sqrt{\frac{\mu_♁}{R_♁}}, \quad v_{II} = \sqrt{\frac{2\mu_♁}{R_♁}} = \sqrt{2}\, v_I . \qquad (5.4)$$

When $\mu_♁$ = 398,600 km^3/sec^2 (see problem 4.6), v_I = 7.91 km/sec, and v_{II} = 11.19 km/sec.

When calculating v_I and v_{II} we can use other formulas. Actually, by comparing Eqs. (1.10) and (4.9) we conclude that $\mu_♁ = fM_♁ = gR_♁$, whence

$$v_I = \sqrt{gR_♁}, \quad v_{II} = \sqrt{2gR_♁}. \qquad (5.5)$$

In the case when a body is launched at an arbitrary distance from the center ($r_0 > R_♁$), we have

$$v_{l.ci} = \sqrt{\frac{\mu_♁}{r_0}}, \quad v_{l.par} = \sqrt{\frac{2\mu_♁}{r_0}}. \qquad (5.6)$$

By combining Eqs. (5.6) and (5.5), we can set up other formulas relating velocities for the launch of a body at altitude H above the Earth's surface:

$$v_{l.ci} = \sqrt{\frac{\mu_♁}{r_0}} = \sqrt{\frac{gR_♁^2}{R_♁+H}} = v_I\sqrt{\frac{R_♁}{R_♁+H}} = 7.91\sqrt{\frac{R_♁}{R_♁+H}},$$
$$v_{l.par} = \sqrt{\frac{2\mu_♁}{r_0}} = \sqrt{\frac{2gR_♁^2}{R_♁+H}} = v_{II}\sqrt{\frac{R_♁}{R_♁+H}} = 11.19\sqrt{\frac{R_♁}{R_♁+H}}. \qquad (5.7)$$

Using Eq. (5.6), we can write /35

$$e = \sqrt{1 + \frac{c^2}{\mu^2}(v_0^2 - v_{l.par}^2)},$$

[2] The terms local circular and local parabolic velocities mean that by these formulas we can calculate the theoretical values of these velocities for any given r. The actual velocity of a point may or may not coincide with these values. In the case when these velocities are the actual velocities in circular or parabolic trajectories, we will use the terms circular or parabolic velocity. In place of the term second escape velocity we can use the term escape velocity.

on the basis of which, allowing for all the above-presented relations, we can set up the following table (see Table 3).

Table 3

Circle $e = 0$	$\tilde{h} = -\mu^2/c^2 < 0$	$v_0 = \sqrt{\mu/r_0} = v_{1.ci}$
Ellipse $e < 1$	$\tilde{h} < 0$	$v_0 = \sqrt{2\mu/r_0}$
Parabola $e = 1$	$\tilde{h} = 0$	$v_0 = \sqrt{2\mu/r_0} = v_{1.par}$
Hyperbola $e > 1$	$\tilde{h} > 0$	$v_0 > \sqrt{2\mu/r_0}$

Problem 5.3. The second Soviet space rocket had a velocity of 2.31 km/sec at the distance 320,000 km from the Earth's center. Considering that motion occurred in a conic section, determine the shape of the trajectory. What velocity did the rocket have at the altitude H = 230 km from the Earth's surface?

Solution. Let us use Eq. (5.2) determining the invariancy of the following quantities for this trajectory in the calculations:

$$\tilde{h} = \left(v^2 - \frac{2\mu_\delta}{r}\right)_{r=320000} = \left(v^2 - \frac{2\mu_\delta}{r}\right)_{r=6600}.$$

Let us determine the constant of energy from the velocity:

$$\tilde{h} = \left(v^2 - \frac{2\mu_\delta}{r}\right)_{r=320000} = 2.84\ \frac{\text{km}^2}{\text{sec}^2} > 0,$$

whence it follows that the trajectory is a hyperbola. Further, we can determine the velocity of the rocket at the altitude H = 230 km:

$$v\Big|_{r=6600} = \sqrt{h + \frac{2\mu_\delta}{R_\delta + H}} = 11.12\ \frac{\text{km}}{\text{sec}}.$$

For comparison, we can calculate the local parabolic velocity at this altitude:

$$v_{1.par} = \sqrt{2\mu_\delta/(R_\delta + H)} = 10.99\ \frac{\text{km}}{\text{sec}}.$$

The slight difference between the actual velocity and the theoretical parabolic velocity enables us to state that the hyperbolic trajectory is nearly the same as a parabola. /36

Problem 5.4. Determine the first v_I and second v_{II} escape velocities for the Moon, Venus, Mars, and Jupiter, by using the gravitational parameters and radii of bodies listed in Table 4.

TABLE 4

Planet	μ, km^3/sec^2	R/km
Moon	4903	1740
Venus	324,460	6100
Mars	42,650	3407
Jupiter	$1.25 \cdot 10^8$	69,900

Solution. By calculating velocities using the formulas

$$v_I = \sqrt{\frac{\mu}{R}}, \quad \text{and} \quad v_I = \sqrt{2\mu/R} = \sqrt{2}\, v_I,$$

we get the following velocities (Table 5).

TABLE 5

Planet	v_I, km/sec	v_{II}, km/sec
Moon	1.68	2.37
Venus	7.29	10.28
Mars	3.54	4.99
Jupiter	42.30	59.64

Problem 5.5. To the first approximation, the orbit of the Moon can be assumed to be a circle with radius r = 384,400 km ≈ ≈ 60.4 $R_{♁}$. Determine the local circular $v_{1.ci}$ and local parabolic $v_{1.par}$ velocities of a body relative to the Earth at this distance.

Solution. From Eqs. (5.7) we have

$$v_{1ci} = v_1 \sqrt{\frac{R_\delta}{60.4 R_\delta}} = 7.91 \sqrt{\frac{1}{60.4}} = 1.02 \text{ km/sec},$$

$$v_{1.par} = \sqrt{2}\, v_{1ci} = 1.44 \text{ km/sec}.$$

Problem 5.6. Determine the local circular and local parabolic velocities of a point moving around the Sun in the Earth's circular orbit ($r = \alpha = 149.6 \cdot 10^6$ km).

Solution. The velocities of the solar satellite can be calculated from expressions analogous to Eqs. (5.6):

$$v_{1ci} = \sqrt{\frac{\mu_\odot}{r}} = \sqrt{\frac{1327 \cdot 10^8}{149.6 \cdot 10^6}} = 29.78 \text{ km/sec},$$

$$v_{1.par} = \sqrt{2}\, v_{1ci} = 42.11 \text{ km/sec}.$$

Problem 5.7. Knowing the expressions for the radius-vector of a point in elliptical motion about an attracting center,

$$\bar{r} = p\bar{r}^0/(1 + e\cos\varphi) = \alpha\,(1 - e\cos E)\,\bar{r}^0,$$

where the unit vector $\bar{r}^0$ is the unit vector in the radial direction, E is the eccentric anomaly, and ϕ is the true anomaly, find the expressions for the vector of orbital velocity of this point in orbital and inertial coordinate systems.

Solution I (for the orbital system of axes). Any system of axes one of whose planes coincides with the orbital plane is called an orbital system of axes. In this case we can use as the orbital system a moving system of polar axes $\bar{r}^0$ and $\bar{\phi}^0$ (Fig. 7 a), all the more so because the components of the orbital velocity $\bar{v}$ in the radial and transversal directions are already determined (see problem 3.8). Using Eqs. (3.9), that is,

$$v_r = \frac{c}{p} e \sin\varphi, \quad v_\varphi = \frac{c}{p}(1 + e\cos\varphi), \quad \text{and} \quad c = \sqrt{\mu p},$$

let us write the expression for the vector of orbital velocity in the orbital system of axes:

$$\bar{v} = \bar{v}_r + \bar{v}_\varphi = \sqrt{\frac{\mu}{p}}\left[e\sin\varphi\,\bar{r}^0 + (1 + e\cos\varphi)\,\bar{\varphi}^0\right].$$

Solution II (for an inertial system of axes). The same system of polar axes $\bar{r}^0$ and $\bar{\phi}^0$ placed at a point which for this problem we can assume to be fixed can be considered an inertial system of axes. It is convenient to select the pericenter of the orbit as this point (Fig. 7 a). /38

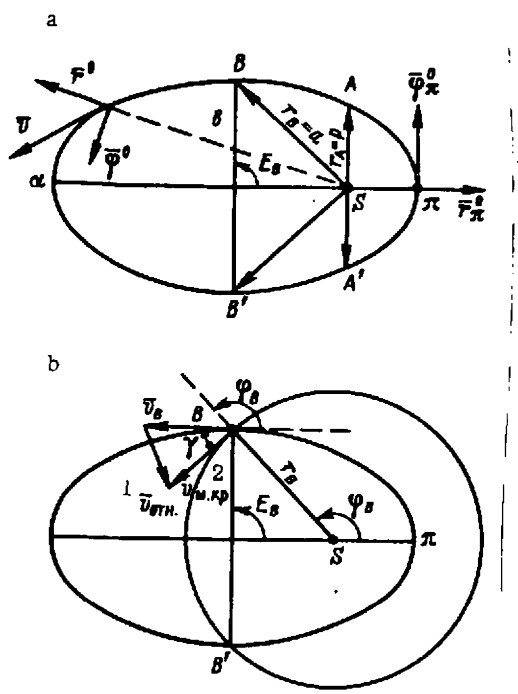

Fig. 7

Key: 1. $\bar{v}_{rel}$
2. $v_{l.ci}$

Let us write the formulas for transforming the axes of the moving system $\bar{r}^0$ and $\bar{\phi}^0$ into the axes of the fixed system $\bar{r}^0$ and $\bar{\phi}^0$:

$$\bar{r}^0 = \bar{r}_\pi^0 \cos\varphi + \bar{\varphi}_\pi^0 \sin\varphi,$$
$$\bar{\varphi}^0 = \bar{\varphi}_\pi^0 \cos\varphi - \bar{r}_\pi^0 \sin\varphi,$$

where ϕ is the true anomaly of point M, that is, at the origin of the moving system. By substituting the formulas of transformation into the formula velocity obtained above, we get the expression for the orbital velocity framed in the inertial system of axes:

$$v = \sqrt{\frac{\mu}{p}} \left[e \sin\varphi \left(\bar{r}_\pi^0 \cos\varphi + \bar{\varphi}_\pi^0 \sin\varphi \right) + (1 + e\cos\varphi) \times \right.$$
$$\left. \times \left(\bar{\varphi}_\pi^0 \cos\varphi - \bar{r}_\pi^0 \sin\varphi \right) \right] = \sqrt{\frac{\mu}{p}} \left[-\bar{r}_\pi^0 \sin\varphi + \bar{\varphi}_\pi^0 (e + \cos\varphi) \right].$$

To transform, use the relations of elliptical motion (2.4) and (2.5), whence

$$\sin\varphi = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E},$$
$$\cos\varphi = \frac{\cos E - e}{1 - e\cos E},$$
$$e + \cos\varphi = \frac{(1-e^2)\cos E}{1 - e\cos E},$$

so that /39

$$\bar{v} = \sqrt{\frac{\mu}{p}} \left[-\frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}\, \bar{r}_\pi^0 + \frac{(1-e^2)\cos E}{1 - e\cos E}\, \bar{\varphi}_\pi^0 \right] =$$
$$= \sqrt{\frac{\mu}{p}} \frac{\sqrt{1-e^2}}{1 - e\cos E} \left[-\sin E\, \bar{r}_\pi^0 + \sqrt{1-e^2}\cos E\, \bar{\varphi}_\pi^0 \right].$$

When $p = \alpha(1 - e^2)$, we can write finally

$$\bar{v} = \sqrt{\frac{\mu}{\alpha}} \cdot \frac{1}{1 - e\cos E}\left[-\sin E\, \bar{r}^0_\pi + \sqrt{1-e^2}\cos E\, \bar{\varphi}^0_\pi\right].$$

$$v = \sqrt{\frac{\mu}{\alpha}} \cdot \frac{\sqrt{\sin^2 E + (1-e^2)\cos^2 E}}{1 - e\cos E} = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{1+e\cos E}{1-e\cos E}}.$$

For the particular cases of the pericenter ($E = 0$) and apocenter ($E = 180°$), we have

$$v_\pi = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{1+e}{1-e}}, \quad v_\alpha = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{1-e}{1+e}}.$$

Note that the latter formulas can be obtained directly from relations (3.10) (see next problem).

Problem 5.8. Derive formulas relating velocities at the apsides v_π and v_α, energy constant $\tilde{h}$, and total orbital velocity v with semi-major axis α of the elliptical orbit.

Solution. In problem 3.8, based on the integral of areas the following formulas were derived:

$$v_\pi = c(1+e)/p, \quad v_\alpha = c(1-e)/p,$$

and in problem 4.3 it was established that $\mu = c^2/p$, so that $c = \sqrt{\mu p}$ and $c/p = \sqrt{\mu/p}$, whence

$$v_\pi = \sqrt{\frac{\mu}{p}}(1+e), \quad v_\alpha = \sqrt{\frac{\mu}{p}}(1-e). \tag{5.8}$$

In addition, for elliptical motion $p = \alpha(1-e^2)$, therefore

$$v_\pi = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{1+e}{1-e}}, \quad v_\alpha = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{1-e}{1+e}}, \tag{5.9}$$

and the formula relating velocities at the apsides is as follows:

$$v_\alpha = v_\pi \frac{1-e}{1+e}, \quad v_\pi = v_\alpha \frac{1+e}{1-e}, \tag{5.10}$$ /40

$$v_{min} = v_{max}\frac{1-e}{1+e}, \quad v_{max} = v_{min}\frac{1+e}{1-e}. \tag{5.10'}$$

From problem 5.1 it follows that the constant of energy $\tilde{h}$ is defined by the formula

$$\tilde{h} = v^2 - \frac{2\mu}{r} = v_\pi^2 - \frac{2\mu}{r_\pi} = \text{const.} \tag{5.11}$$

Substituting instead of v_π the expression from (5.9) corresponding to it, and instead of r_π, its value $r_\pi = \alpha(1 - e)$, we get

$$\tilde{h} = \frac{\mu(1+e)}{\alpha(1-e)} - \frac{2\mu}{\alpha(1-e)} = -\frac{\mu(1-e)}{\alpha(1-e)} = -\frac{\mu}{\alpha}. \tag{5.11'}$$

From Eqs. (5.11) and (5.11') there follows the formula relating v and α:

$$v^2 = \tilde{h} + \frac{2\mu}{r} = \mu\left(\frac{2}{r} - \frac{1}{\alpha}\right). \tag{5.12}$$

Formula (5.12) is also customarily called the integral of energy.

Problem 5.9. Express the velocity at any point of an elliptical orbit in terms of the eccentric anomaly.

Solution. Let us substitute into Eq. (5.12) the formula for the radius-vector expressed in terms of the eccentric anomaly (see problem 2.1), $r = \alpha(1 - e \cos E)$. As a result we get

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{\alpha}\right)} = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{2\alpha - r}{r}} =$$

$$= \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{2\alpha - \alpha(1 - e\cos E)}{\alpha(1 - e\cos E)}} = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{1 + e\cos E}{1 - e\cos E}}.$$

The same result was derived in problem 5.7 in another way.

Problem 5.10. Determine the total orbital energy of a point moving under the influence of a gravitational force as a function of the semi-major axis of elliptical orbit α.

Solution. Let us write the equation of the energy balance of the point:

$$h = T + V = T - U = \frac{mv^2}{2} - \frac{\mu m}{r} = \frac{m}{2}\left(v^2 - \frac{2\mu}{r}\right) = \frac{m}{2}\tilde{h}. \tag{5.13}$$

By substituting into (5.13) instead of the constant $\tilde{h}$, its value expressed in terms of α, that is, Eq. (5.11'), we find /41

$$h = \frac{m}{2}\tilde{h} = -\frac{\mu m}{2\alpha} = \text{const.} \tag{5.13'}$$

Thus, the total energy of a point moving in an elliptical orbit with semi-axis does not depend on the radius-vector. At the pericenter the potential enerby $V = -\mu m/r$ is at a minimum (the radius-vector r is at a minimum, and the potential $U = \mu m/r$ is at a maximum). Therefore the kinetic energy $T = mv^2/2$, and this means also the orbital velocity v is at a maximum. But at the apocenter the potential energy is at a maximum and the kinetic energy and velocity are at a minimum.

If two satellites move around a planet in circular orbits, the one that is farthest from the planet has the greatest orbital energy.

Problem 5.11. Set up a table of energy balance for four characteristic points of an ellipse by using the integral of energy.

Solution. The following six points (Fig. 7 a) are called the characteristic points of an ellipse: the points of intersection of the ellipse with its semi-major axis, or apsides, that is, pericenter π ($\phi = E = 90°$) and apocenter α ($\phi = E = 180°$), points A ($\phi = 90°$) and A' ($\phi = 180°$), whose radius-vector is the focal parameter of the ellipse $r_A = r_{A'} = p = \alpha(1 - e \cos E_A) = \alpha(1 - e^2)$, and the eccentric anomaly $E_A = E_{A'} = \arccos e$, and the points of intersection of the ellipse with its semi-minor axis B and B', whose radius-vectors are equal to the semi-major axis based on the relations

$$r_B = r_{B'} = \sqrt{(\alpha e)^2 + b^2} = \sqrt{\alpha^2 e^2 + \alpha^2(1-e^2)} = \alpha.$$

For the two last points the corresponding polar angle (true anomaly) $\phi = \arccos(-e)$, since $\sin\phi = b/\alpha = \sqrt{1-e^2}$; $\cos\phi = -e$, and the eccentric anomalies $E_B = 90°$, $E_{B'} = 270°$.

Using the energy relations (5.12), (5.13), and (5.13') for four (of six) characteristic points of the ellipse, let us determine the quantities

$$T = \frac{mv^2}{2} = \frac{\mu m}{2}\left(\frac{2}{r} - \frac{1}{\alpha}\right), \quad V = -\frac{\mu m}{r}, \quad \text{and} \quad h = -\frac{\mu m}{2\alpha}$$

and let us set up the following table.

TABLE 6

Characteristic points of ellipse	φ	E	r	$T/\mu m$	$V/\mu m$	$h/\mu m$
π	0	0	$a(1-e)$	$\frac{1+e}{2a(1-e)}$	$-\frac{1}{a(1-e)}$	$-\frac{1}{2a}$
A	90°	arccos e	$a(1-e^2)$	$\frac{1+e^2}{2a(1-e^2)}$	$-\frac{1}{a(1-e^2)}$	$-\frac{1}{2a}$
B	arccos(-e)	90°	a	$\frac{1}{2a}$	$-\frac{1}{a}$	$-\frac{1}{2a}$
α	180°	180°	$a(1+e)$	$\frac{1-e}{2a(1+e)}$	$-\frac{1}{a(1+e)}$	$-\frac{1}{2a}$

/42

Problem 5.12. Prove that when a satellite moving in an elliptical orbit transits the end of its semi-minor axis B its velocity is equal in absolute magnitude to the local circular velocity $v_{1.ci}$.

Solution. It was proved in problem 5.11 that the radius-vector of the end point of the semi-minor axis of an ellipse is equal to the semi-major axis ($r_B = a$) (Fig. 7 a), therefore the orbital velocity of this point can be calculated by Eq. (5.12),

$$v_B = \sqrt{\mu(2/r_B - 1/a)} = \sqrt{\mu/a},$$

and this expression coincides with the expression for the local circular velocity at distance $r = a$ from the attracting center.

Fig. 7 b shows the elliptical and circular orbits corresponding to this case. These orbits have the same total energy, since the energy constant $\tilde{h} = -\mu/a = -\mu/r_B$ is the same for them. Note that the equality of velocities can derive directly from the equality of the kinetic energies of these two orbits, since total orbital energy h and the potential energy $V = -\mu m/r$ for them are also the same. From Fig. 7 b it also follows that the velocity vectors $\bar{v}_{1.ci}$ and $\bar{v}_B$ form an equilateral vector triangle.

Problem 5.13. Find in an elliptical orbit the points where the velocity is equal to the mean geometrical velocity at the pericenter and at the apocenter.

Solution. In problems 5.7 and 5.8 we found the formulas for velocities at the apsides:

$$v_\pi = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}}, \text{ and } v_\alpha = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1-e}{1+e}}.$$

The geometric mean of these velocities is of the form /43

$$v^* = \sqrt{v_\pi v_\alpha} = \sqrt{\mu/a}.$$

However, in problem 5.12 it was shown that this velocity is attained when a satellite moving in an elliptical orbit transits the end point of its semi-minor axis where $r_B = a$. Thus, both points desired lie at the end points of the semi-minor axis of the orbit ($E = 90°$ and $270°$).

Problem 5.14. When a satellite moving in an elliptical orbit transits the end point of its semi-minor axis B (Fig. 7 b), its velocity v_B is equal in absolute magnitude to the local circular velocity $v_{1.ci}$, so that the vectors $\bar{v}_B$ and $\bar{v}_{1.ci}$ form an equilateral vector triangle (see problem 5.12). Determine the relative velocity $\bar{v}_{rel} = \bar{v}_{1.ci} - \bar{v}_B$ and express it in terms of the orbital eccentricity.

Solution. The vector of relative velocity is the closing leg of the equilateral vector triangle, so that

$$v_{rel}^2 = v_{1.ci}^2 + v_B^2 - 2\, v_{1.ci} v_B \cos(\bar{v}_{1.ci} \hat{,} \bar{v}_B),$$

where $v_{1.ci} = v_B$ and $\cos(\bar{v}_{1.ci} \hat{,} \bar{v}_B) = \cos(360° - \gamma) = \cos\gamma =$

$= \cos(\phi_B - 90°) = \cos(90° - \phi_B) = \sin\phi_B$.

It was established in problem 5.11 that the true anomaly of the end point of the semi-minor axis $\varphi_B = \arccos(-e)$, so that

$$\sin\varphi_B = \sin[\arccos(-e)] = \sqrt{1-e^2}.$$

Substituting this expression into the velocity formula, we have

$$v_{rel}^2 = 2\, v_{1.ci}^2 \left(1-\sqrt{1-e^2}\right), \quad r_{rel} = \sqrt{2}\, v_{1.ci}\left[1-(1-e^2)^{1/2}\right]^{1/2}.$$

Expanding in a series, we get an approximate formula of relative velocity:

$$v_{rel} \approx v_{1.ci}\, e\left(1+\frac{e^2}{8}\right).$$

The angle at the base of the vector triangle is determined by the formula

$$\gamma = \varphi_s - 90^\circ = \varphi_s - E_s .$$

Problem 5.15. At some point of an orbit at distance r from the attracting planet, its satellite has the velocity v. The local parabolic velocity at this point is $v_{l.par}$. Determine the semi-major axis of the elliptical orbit α. /44

Solution. The integral of energy can be written in terms of a local parabolic velocity as follows:

$$\tilde{h} = v^2 - \frac{2\mu_{pl}}{r} = v^2 - v^2_{l.par},$$

where $h = -\mu_{pl}/\alpha$, so that

$$\alpha = \frac{\mu_{pl}}{v^2_{l.par} - v^2} .$$

If μ_{pl} is unknown, it can be determined from the known r and $v_{l.par}$: $\mu_{pl} = \frac{1}{2} v^2_{l.par} r$.

Problem 5.16. Upon transiting across the pericenter, the satellite of a planet has the velocity v_π. Its local circular velocity at the pericenter is $v_{l.ci\ \pi}$. Determine the eccentricity of the elliptical orbit. Solve this same problem for the case of a satellite transiting the apocenter.

Solution. By writing the integral of energy $v^2 = \mu(2/r - 1/\alpha)$ and using the formula of the radius-vector of the pericenter, we can state

$$v_\pi^2 = \frac{2\mu}{r_\pi} - \frac{\mu}{\alpha} = \frac{2\mu}{r_\pi} - \frac{\mu(1-e)}{r_\pi} = \frac{\mu}{r_\pi}(1+e) = v^2_{l.ci}\ (1 + e)$$

Whence

$$e = \frac{v_\pi^2}{v^2_{l.ci\ \pi}} - 1 .$$

Since for an elliptical orbit $0 < e < 1$, then we have

$$1 < \frac{v_\pi^2}{v_{1.ci\pi}^2} < 2, \text{ i.e. } v_{1.ci\pi} < v_\pi < v_{1.ci\pi}\sqrt{2} = v_{1.par\pi}.$$

Thus, to determine the eccentricity it is sufficient to know the v_π and $v_{1.ci\ \pi}$ at the pericenter.

We can easily show that the formula obtained above for the eccentricity can be derived directly from Eq. (5.3):

$$e = \sqrt{1 + \frac{c^2}{\mu^2}\tilde{h}} = \sqrt{1 + \frac{(v_\pi r_\pi)^2}{\mu^2}\left(v_\pi^2 - \frac{2\mu}{r_\pi}\right)} =$$

$$= \sqrt{\left(1 - \frac{v_\pi^2}{v_{1.ci\pi}^2}\right)^2} = \pm\left(1 - \frac{v_\pi^2}{v_{1.ci\pi}^2}\right).$$

Since $v_\pi > v_{1.ci\ \pi}$, the expression in the parentheses is negative, /45
and therefore to ensure $e > 0$ we must select the formula with a "minus" sign, then we have

$$e = \frac{v_\pi^2}{v_{1.ci\pi}^2} - 1.$$

When the satellite transits the apocenter, we have

$$v_\alpha^2 = \frac{2\mu}{r_\alpha} - \frac{\mu}{a} = \frac{2\mu}{r_\alpha} - \frac{\mu(1+e)}{r_\alpha} = v_{1.ci\alpha}^2(1-e),$$

whence

$$e = 1 - \frac{v_\alpha^2}{v_{1.ci\alpha}^2},$$

so that $0 < \frac{v_\alpha^2}{v_{1.ci\alpha}^2} < 1$, that is, $0 < v_\alpha < v_{1.ci\alpha}$.

Problem 5.17. The greatest distance of Sputnik-3 from the surface of the Earth H = 1880 km, and the smallest h - 230 km (Fig. 8). Determine its velocities at the apogee v_α and at the perigee v_π.

Solution. From Fig. 8 the geometrical relations

$$2a = 2R_\delta + H + h,$$

$$a = R_\delta + \frac{1}{2}(H+h) = 6370 + $$
$$+ \frac{1}{2}(1880 + 230) = 7425 \text{ km}.$$

are obvious. The eccentricity can be determined by Eq. (2.13'):

$$e = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi} = \frac{(R_\delta + H) - (R_\delta + h)}{(R_\delta + H) + (R_\delta + h)} =$$

$$= \frac{H - h}{2R_\delta + H + h} = \frac{1650}{14{,}850} = 0.11.$$

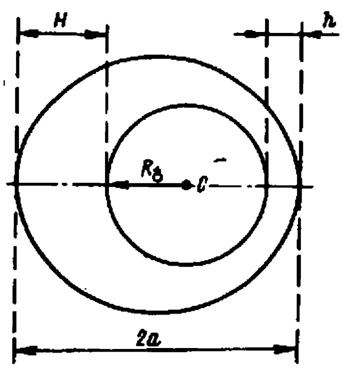

Fig. 8

To compute the velocities at the apsides, apply Eqs. (5.9) and (5.10):

$$v_\alpha = \sqrt{\frac{\mu_\delta}{a}}\sqrt{\frac{1-e}{1+e}} = \sqrt{\frac{398{,}600}{7425}}\sqrt{\frac{1-0.11}{1+0.11}} = 6.56 \ \frac{\text{km}}{\text{sec}},$$

$$v_\pi = v_\alpha \frac{1+e}{1-e} = 6.56 \ \frac{1.11}{0.89} = 8.18 \ \frac{\text{km}}{\text{sec}}.$$

Thus, the velocity at the perigee /46
is 25 percent greater than at the apogee.

Problem 5.18. Calculate the velocity of a satellite whose orbit has a large eccentricity at the perigee v_π and at the apogee v_α. The largest distance of the satellite from the Earth's surface H, and the shortest distance h are 42,450 and 252 km, respectively.

Solution. Calculate a and e, as in problem 5.17:

$$a = R_\delta + \frac{1}{2}(H+h) = 6370 + \frac{1}{2}(42{,}450 + 252) = 27{,}721 \text{ km},$$

$$e = \frac{H-h}{2R_\delta + H + h} = \frac{42{,}198}{55{,}442} = 0.76,$$

$$v_\alpha = \sqrt{\frac{\mu_\delta}{a}}\sqrt{\frac{1-e}{1+e}} = \sqrt{\frac{398{,}600}{27{,}721}}\sqrt{\frac{1-0.76}{1+0.76}} = 1.40 \text{ km/sec},$$

$$v_\pi = v_\alpha \frac{1+e}{1-e} = 1.40 \ \frac{1+0.76}{1-0.76} = 10.27 \text{ km/sec}.$$

In this case the velocity of the perigee is more than seven times the apogee velocity.

Problem 5.19. Find the largest and smallest velocities of the Earth around the Sun. Compare the velocities obtained with the Earth's mean heliocentric velocity.

Solution. In problem 2.3 the larges (aphelion) r_α and smallest (perihelion) r_π distances of Earth from Sun were calculated: $r_\pi = 147.1 \cdot 10^6$ km, and $r_\alpha = 152.1 \cdot 10^6$ km, and it was also shown that the arithmetic mean of these distances is equal to the semi-major axis: $r_{av} = 1/2\ (r_\pi + r_\alpha) = 149.6 \cdot 10^6$ km. In problem 2.5 the mean integral value of the heliocentric distance of the Earth $[r]_t$ was determined. Using these values, we can calculate the quantities we seek. Thus, the orbital velocity of the Earth reaches its largest and smallest values at the perihelion and aphelion, respectively:

$$
\begin{aligned}
v_\pi &= \sqrt{\mu_\odot\left(\frac{2}{r_\pi} - \frac{1}{a}\right)} = \sqrt{1327 \cdot 10^8\left(\frac{2}{147.1 \cdot 10^6} - \frac{1}{149.6 \cdot 10^6}\right)} = 30.28 \text{ km/sec},\\
v_\alpha &= \sqrt{\mu_\odot\left(\frac{2}{r_\alpha} - \frac{1}{a}\right)} = \sqrt{1327 \cdot 10^8\left(\frac{2}{152.1 \cdot 10^6} - \frac{1}{149.6 \cdot 10^6}\right)} = 27.53 \text{ km/sec}.
\end{aligned}
\tag{5.14}
$$

(we can find v_α also by Eq. (3.6) from problem 3.4). The arith- /47
metic mean of these velocities is:

$$
v_{av} = \frac{1}{2}(v_\pi + v_\alpha) = \frac{1}{2}\left[\sqrt{\mu_\odot\left(\frac{2}{r_\pi} - \frac{1}{a}\right)} + \sqrt{\mu_\odot\left(\frac{2}{r_\alpha} - \frac{1}{a}\right)}\right] = 28.90 \ \frac{\text{km}}{\text{sec}}.
$$

Now let us determine what we must adopt as the mean integral velocity [v]. As already pointed out in problem 2.5, averaging can be carried out with respect to any variable used in the theory of elliptical motion: with respect to time t, mean anomaly M, eccentric anomaly E, or true anomaly ϕ (see problem 2.1).

Let us first look at time averaging, when

$$
[v]_t = \frac{1}{T}\int_0^T v\,dt,
$$

where T is the theory of the Earth's revolution around the Sun. Adopt as the new variable of integration the eccentric anomaly E, since in problem 5.9 we derived a convenient formula relating v to E:

$$
v = \sqrt{\frac{\mu}{a}}\ \sqrt{\frac{1 + e\cos E}{1 - e\cos E}}.
$$

From Kepler's equation (see problem 2.5) we have $E - e\sin E =$

$$= n(t-T) = M, \quad dE - e\cos E\, dE = n\,dt = dM, \qquad dt =$$
$$= (1 - e\cos E)\, dE/n,$$

so that we finally get the elliptical integral of the second kind

$$[v]_t = \frac{1}{Tn}\sqrt{\frac{\mu}{a}}\int_0^{2\pi}\sqrt{1-e^2\cos^2 E}\; dE = \frac{1}{Tn}\sqrt{\frac{\mu}{a}}\int_0^{2\pi}\Big(1 - \frac{1}{2}e^2\cos^2 E -$$
$$- \frac{1}{8}e^4\cos^4 E + \ldots\Big)dE = \frac{1}{Tn}\sqrt{\frac{\mu}{a}}\Big(E - \frac{1}{4}e^2 E - \frac{1}{8}e^2\sin 2E + \frac{3}{64}e^4 E + \ldots\Big)\Big|_0^{2\pi} =$$
$$= \frac{2\pi}{Tn}\sqrt{\frac{\mu}{a}}\Big(1 - \frac{1}{4}e^2 + \frac{3}{64}e^4 + \ldots\Big).$$

Since $T = 2\pi/n$, we find

$$[v]_t = \sqrt{\frac{\mu}{a}}\Big(1 - \frac{1}{4}e^2 + \frac{3}{64}e^4 + \ldots\Big).$$

We can easily show that the same result can be obtained by averaging over the mean anomaly M, which is an analog of time in elliptical motion. Actually, since $n\,dt = dM = (1 - e\cos E)\,dE$, /48

$$[v]_M = \frac{1}{2\pi}\int_0^{2\pi} v\,dM = \frac{1}{2\pi}\sqrt{\frac{\mu}{a}}\int_0^{2\pi}\sqrt{1-e^2\cos E}\,dE = \sqrt{\frac{\mu}{a}}\Big(1 -$$
$$- \frac{1}{4}e^2 + \frac{3}{64}e^4 + \ldots\Big).$$

On the other hand, averaging over the eccentric anomaly E leads to a different result:

$$[v]_E = \frac{1}{2\pi}\int_0^{2\pi} v\,dE = \frac{1}{2\pi}\sqrt{\frac{\mu}{a}}\int_0^{2\pi}\sqrt{\frac{1+e\cos E}{1-e\cos E}}\,dE =$$
$$= \frac{1}{2\pi}\sqrt{\frac{\mu}{a}}\int_0^{2\pi}\Big((1+e\cos E) + \frac{1}{2}e^2\cos^2 E + \ldots\Big)dE = \sqrt{\frac{\mu}{a}}\Big(1 + \frac{1}{4}e^2 + \ldots\Big).$$

Averaging with respect to the true anomaly ϕ, we have (see problem 2.5, where $d\phi = a/r\sqrt{1-e^2}\; dE$)

$$[v]_\varphi = \frac{1}{2\pi}\int_0^{2\pi} v\,d\varphi = \frac{1}{2\pi}\sqrt{\frac{\mu}{\alpha}}\int_0^{2\pi}\sqrt{\frac{1+e\cos E}{1-e\cos E}}\,\frac{\sqrt{1-e^2}\,dE}{(1-e\cos E)} =$$

$$= \frac{1}{2\pi}\sqrt{\frac{\mu}{\alpha}}\sqrt{1-e^2}\int_0^{2\pi}\left(1+2e\cos E+\frac{5}{2}e^2\cos^2 E+\ldots\right)dE =$$

$$= \sqrt{\frac{\mu}{\alpha}}\left(1-\frac{1}{2}e^2+\ldots\right)\left(1+\frac{5}{4}e^2+\ldots\right) = \sqrt{\frac{\mu}{\alpha}}\left(1+\frac{3}{4}e^2+\ldots\right).$$

Thus, it is difficult to give an exact determination of the Earth's mean velocity, however considering the smallness of the eccentricity of the Earth's orbit ($e = 0.01678$), with adequate accuracy we can adopt as the mean Earth velocity the first term in the expansions obtained above, that is, $[v] = \sqrt{\mu_\odot/\alpha}$. In other words, the mean Earth velocity (or the velocity of another planet) can be assumed, with adequate accuracy, to be the velocity that it would have if it moved uniformly in a circle with a radius equal to the semi-major axis of its orbit. This definition of the mean velocity of the Earth coincides with the definition of its heliocentric circular velocity as the Earth transits the end point of /49
its semi-minor axis at a distance α from the Sun (see problems 5.12 and 5.13). Moreover, we note that the quantity $[v] = \sqrt{\mu_\odot/\alpha}$ is simultaneously the circular velocity of a point at distance $r_{av} = [r]_E = \alpha$ from the Sun. On the other hand, the mean velocity can be defined as the heliocentric circular velocity of a point at distance $[r]_t = [r]_M$:

$$[v] = \sqrt{\frac{\mu_\odot}{[r]_t}} = \sqrt{\frac{\mu_\odot}{\alpha\left(1+\frac{1}{2}e^2\right)}} = \sqrt{\frac{\mu_\odot}{\alpha}}\quad\left(1-\frac{1}{4}e^2+\frac{3}{32}e^4+\ldots\right).$$

For small e, as before we can use the definition of the mean velocity presented above.

Thus, by agreeing to call the quantity $[v] = \sqrt{\mu_\odot/\alpha}$ the mean Earth velocity, we note that it does not coincide with the arithmetic mean of the velocity v_π and v_α. To derive the formula relating $[v]$ with v_ρ and v_α, let us substitute in (5.14) $[v]^2$ or $\mu_\odot/\alpha$:

$$[v]^2 = \frac{2\mu_\odot}{r_\pi} - v_\pi^2 = \frac{2\mu_\odot}{r_\alpha} - v_\alpha^2. \tag{5.15}$$

We can set up simpler formulas of the relationship for small eccentricities. Actually, from (5.14) we have

$$v_\pi^2 = \frac{\mu_\delta}{a}\cdot\frac{2a - r_\pi}{r_\pi} = \frac{\mu_\delta}{a}\cdot\frac{r_\alpha}{r_\pi} = [v]\frac{1+e}{1-e},$$
$$v_\alpha^2 = \frac{\mu_\delta}{a}\cdot\frac{2a - r_\alpha}{r_\alpha} = \frac{\mu_\delta}{a}\cdot\frac{r_\pi}{r_\alpha} = [v]\frac{1-e}{1+e}. \qquad (5.16)$$

Hence, for small e we have

$$v_\pi^2 \approx [v]^2(1+2e), \quad v_\alpha^2 \approx [v]^2(1-2e),$$
$$v_\pi \approx [v](1+e), \quad v_\alpha \approx [v](1-e), \qquad (5.17)$$
$$[v] \approx v_\pi(1-e) \approx v_\alpha(1+e).$$

Problem 5.20. What minimum initial velocity must be important to a spacecraft moving parallel to the Earth's surface at altitude h = 230 km for it to reach the Moon at its apogee, by moving in an ellipse that is tangent to the Moon's orbit. The semi-major axis of the Moon's orbit $a_\mathbb{C}$ = 384,400 km, and the eccentricity of the Moon's orbit $e_\mathbb{C}$ = 0.055. Solve the same problem for the case of arrival at the Moon when it is at the apogee of the given spacecraft. /50

Solution. Calculate the apogee and perigee distances of the Moon: $r_{\alpha_\mathbb{C}} = a_\mathbb{C}(1+e_\mathbb{C})$ = 405,500 km, and $r_{\pi_\mathbb{C}} = a(1-e_\mathbb{C})$ = 363,000 km.

To reach the Moon situated at the apogee $\alpha_\mathbb{C}$ of its orbit (Fig. 9), the spacecraft must move in ellipse I (or I') tangent to the orbit of the Moon at apogee $\alpha_\mathbb{C}$ and with its perigee at the launch point A (under the condition the initial velocity is horizontal). Here the semi-major axis of orbit I is

$$a_I = \frac{1}{2}(AO + O\alpha_\mathbb{C}) = \frac{1}{2}(r_\pi + r_{\alpha_\mathbb{C}}),$$

where $r_\pi = R_\delta + h$ = 6370 + 230 = 6600 km is the perigee distance of the spacecraft, so that $a_I = \frac{1}{2}$ (6600 + 405,500) = 206,000 km.

Let us find the initial launch velocity:

$$v_\pi = \sqrt{\mu_\delta\left(\frac{2}{r_\pi} - \frac{1}{a_I}\right)} = \sqrt{398{,}600\left(\frac{2}{6600} - \frac{1}{206{,}000}\right)} = 10.90 \text{ km/sec}.$$

Using the equality $r_{\alpha_I} = r_{\alpha_\mathfrak{C}}$, we can determine the eccentricity of the transfer ellipse:

$$e_I = \frac{r_{\alpha_I} - r_\pi}{r_{\alpha_I} + r_\pi} = \frac{r_{\alpha_\mathfrak{C}} - r_\pi}{r_{\alpha_\mathfrak{C}} + r_\pi} = \frac{398900}{412100} = 0.968$$

(the orbit is strongly elliptical).

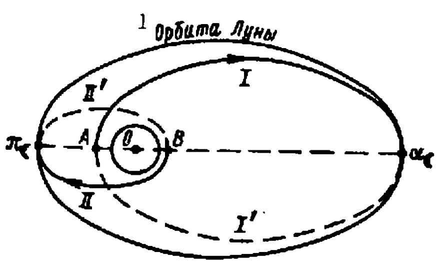

Fig. 9
Key: 1. Orbit of Moon

For the case of the spacecraft reaching the Moon situated at the perigee $\pi_\mathfrak{C}$ of its orbit, we must consider the transfer ellipse II (or II') $(r_{\alpha_{II}} = r_{\pi_\mathfrak{C}})$. In this case we have

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$$a_{II} = \tfrac{1}{2}(BO + O\pi_\mathfrak{C}) = \tfrac{1}{2}(r_\pi + r_{\pi_\mathfrak{C}}) = \tfrac{1}{2}(6600 + 363300) = 184950 \text{ km},$$

$$v_{\pi_{II}} = \sqrt{\mu_\delta\left(\frac{2}{r_\pi} - \frac{1}{a_{II}}\right)} = \sqrt{398600\left(\frac{2}{6600} - \frac{1}{184950}\right)} = 10.89 \text{ km/sec},$$

$$e_{II} = \frac{r_{\alpha_{II}} - r_\pi}{r_{\alpha_{II}} + r_\pi} = \frac{r_{\pi_\mathfrak{C}} - r_\pi}{r_{\pi_\mathfrak{C}} + r_\pi} = \frac{356700}{369900} = 0.991.$$

By comparing the resulting values of v_{π_I} and $v_{\pi_{II}}$ with the corresponding values $v_{1.par}$ for $r_{\pi_I} = r_{\pi_{II}}$, we find $v_{1.par} = \sqrt{\frac{2\mu_\delta}{6600}} = 10.99$ km/sec.

Problem 5.21. Two satellites with equal masses move in the same direction around an attracting center in coplanar orbits, one of which is circular with radius r_0, the other is elliptical with radius-vectors of the pericenter and apocenter r_0 and $8r_0$, respectively. Assuming that the satellites by direct docking connect with each other at the point of tangency of their orbits and move together in further motion, find the radius-vector of the apocenter of their new orbit.

Solution. Since the semi-axis of the circular orbit I (Fig. 10) $a_I = r_0$, and the semi-axis of the elliptical orbit II $a_{II} = 1/2\,(r_0 + 8r_0) = 9/2\,r_0$, the velocities of both satellites at the point of tangency of their orbits, being simultaneously the

pericenter π of orbit II and the new orbit III, are as follows:

$$v_I = \sqrt{\frac{\mu}{r_0}}\ ,\quad v_{\pi_{II}} = \sqrt{\mu\left(\frac{2}{r_0} - \frac{1}{a_{II}}\right)} = \sqrt{\frac{2\mu}{r_0}\left(1 - \frac{1}{9}\right)} = \frac{4}{3}\sqrt{\frac{\mu}{r_0}}\ .$$

Upon satellite docking, velocity change occurs. The velocity of a new satellite, which is the arithmetic mean of the velocity v_I and v_{II} (for the same masses of both satellites), is

$$v_{\pi_{III}} = \frac{1}{2}\left(v_I + v_{\pi_{II}}\right) = \frac{1}{2}\left(1 + \frac{4}{3}\right)\sqrt{\frac{\mu}{r_0}} = \frac{7}{6}\sqrt{\frac{\mu}{r_0}}\ .$$

The semi-major axis of a new orbit is found from the integral of energy

$$v_{\pi_{III}}^2 = \mu\left(\frac{2}{r_0} - \frac{1}{a_{III}}\right),$$

whence

$$\frac{\mu}{a_{III}} = \frac{\mu}{r_0}\left(2 - \frac{49}{36}\right) = \frac{23}{36}\frac{\mu}{r_0} \quad \text{and} \quad a_{III} = \frac{36}{23} r_0 = 1\frac{13}{23} r_0\ .$$

Knowing that $2a_{III} = r_0 + r_{\alpha_{III}}$, we find

$$r_{\alpha_{III}} = \left(\frac{72}{23} - 1\right) r_0 = \frac{49}{23} r_0\ .$$

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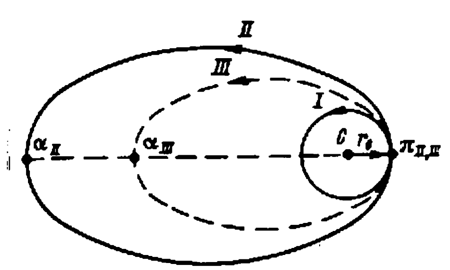

Fig. 10

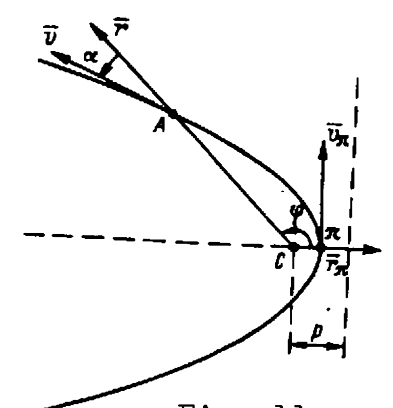

Fig. 11

Problem 5.22. Derive the formula relating the angle $\alpha = (\bar{r}, \hat{\ } \bar{v})$ to the angle of the true anomaly ϕ for the parabolic orbit (Fig. 11).

Solution. Let us use the relations of parabolic motion

$$r = p/(1+\cos\varphi), \quad v = \sqrt{2\mu/r}.$$

In addition, for any conic section the formulas $\sin\alpha = c/rv$, and $c = \sqrt{\mu p}$, are valid, whence

$$\sin\alpha = \frac{\sqrt{\mu p}\,(1+\cos\varphi)\sqrt{p}}{p\sqrt{2\mu(1+\cos\varphi)}} = \sqrt{\frac{1+\cos\varphi}{2}} = \pm\cos\frac{\varphi}{2}.$$

At the pericenter $\varphi = 0$, and $\alpha = \frac{\pi}{2}$. With separation from the pericenter ϕ increases, and $\alpha < \frac{\pi}{2}$ decreases, so that

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right) = +\cos\frac{\varphi}{2}, \quad \alpha = \frac{1}{2}(\pi - \varphi).$$

This result is valid only for parabolic orbits.

Problem 5.23. Establish the dynamic meaning of the integral of energy

$$\tilde{h} = v^2 - \frac{2\mu}{r}$$

for parabolic ($\tilde{h} = 0$) and hyperbolic ($\tilde{h} > 0$) motion.

Solution. Let us rewrite the integral of energy in the form

$$v^2 = \tilde{h} + 2\mu/r,$$

whence

$$\lim_{r\to\infty} v^2 = v_\infty^2 = \lim_{r\to\infty}\left(\tilde{h} + \frac{2\mu}{r}\right) = \tilde{h} = \text{const},$$

so that the constant of energy $\tilde{h}$ (double the total energy of unit mass) can be interpreted as the square of the velocity "at infinity." Here the interval of energy can be written as

$$\tilde{h} = v_\infty^2 = v^2 - v_{1.\mathrm{par}}^2. \qquad (5.18)$$ /53

The quantity v_∞ is also called the excess hyperbolic velocity.

In hyperbolic motion ($\tilde{h} = 0$), the quantity $\tilde{h} = v_\infty^2 = 0$ means that the true velocity coincides wholly with the theoretical value of the local parabolic velocity at a given point of the orbit ($v = v_{l.par}$) and that "at infinity" a stoppage occurs $(v_\infty = \sqrt{\tilde{h}} = 0)$.

In hyperbolic motion $(\tilde{h} > 0)$ the inequality $\tilde{h} = v_\infty^2 > 0$ means that the true hyperbolic velocity exceeds the theoretical value of the local parabolic velocity ($v > v_{l.par}$) and the point will move indefinitely long along the asymptote of the hyperbola at constant velocity $v_\infty = \sqrt{\tilde{h}} = \text{const.}$ the velocity v_∞ is the smallest possible velocity for the given hyperbola. Here the difference between v_∞ and v will tend to zero $(\lim_{r\to\infty} v_{l.par}^2 = 0)$ and the theoretical velocity "at infinity" v_∞ coincides with the actual hyperbolic velocity $(\lim_{r\to\infty} v = v_\infty)$.

Problem 5.24. Determine the velocity at which a meteorite enters the Earth's atmosphere if its velocity "at infinity" $v_\infty = 10.0$ km/sec.

Solution. Let us write the interval of energy in the form of expression (5.18), determining in advance that

$$\tilde{h} = v_\infty^2 = 100 \text{ km}^2/\text{sec}^2 > 0:$$

thus, a hyperbola will be the meteorite's trajectory. To determine the theoretical value of the local parabolic velocity upon entry into the atmosphere, we will assume that entry occurs at an altitude of about 200 km above the Earth's surface so that $v_{l.par} = 11.0$ km/ /sec (see problem 5.3). Let us calculate the entry velocity of the meteorite into the atmosphere:

$$v = \sqrt{v_\infty^2 + v_{l.par}^2} = \sqrt{221} = 14.9 \text{ km/sec}.$$

Problem 5.25. Knowing that for a hyperbola e > 1 and $\alpha < 0$, and that the formulas of elliptical motion can be transformed into the formulas of hyperbolic motion by replacing α with $-|\alpha|$, let us write the formulas of the velocity of a point v and the constant of energy $\tilde{h}$ for hyperbolic motion, bearing in mind the motion along the branch of the hyperbola whose focus is the attracting center C (Fig. 12).

Solution. Using the formulas of elliptical motion /54

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \quad \tilde{h} = -\frac{\mu}{a}$$

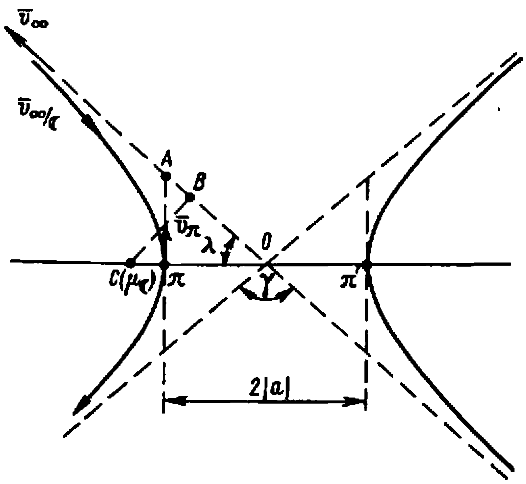

Fig. 12

and replacing α in them by $-|\alpha|$, for the hyperbola we find

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{|a|}\right),\quad \tilde{h} = \frac{\mu}{|a|}. \tag{5.19}$$

We note that the resulting formulas (5.19) enable us again to obtain the results of problem 5.23. Actually,

$$|a| = \frac{\mu}{\tilde{h}} = \frac{\mu}{v_\infty^2},$$

$$v = \sqrt{\mu\left(\frac{2}{r} + \frac{v_\infty^2}{\mu}\right)} = v_\infty\sqrt{\frac{2}{r}\frac{\mu}{v_\infty^2} + 1},$$

whence, by taking the difference $v_\infty^2 = v^2 - v_{1.par}^2$ to be a constrained quantity, we again have $\lim_{r\to\infty} v = v_\infty$.

Problem 5.26. For a spacecraft 500,000 km from the Earth's center, the theoretical velocity "at infinity" $v_\infty = 4.00$ km/sec. Determine the true velocity of the craft v and the length of the semi-major axis of its orbit $|\alpha|$, by first ascertaining in advance the shape of the orbit.

Solution. Determine the constant of energy: $\tilde{h} = v_\infty^2 = 16$ km^2/sec^2 > 0. Now we can conclude that the craft trajectory is a hyperbola (Fig. 12). Let us find the semi-major axis of the hyperbola from the integral of energy /55

$$\tilde{h} = \frac{\mu_\delta}{|a|},$$

so that

$$|a| = \pi O = 398{,}600/16 = 24{,}900 \text{ km}.$$

We can calculate the instantaneous hyperbolic velocity from the formula

$$v^2 = v_\infty^2 + v_{1.par}^2,$$

where

$$v_{1.par} = \sqrt{2\mu_\delta/r} = \sqrt{2\cdot 398{,}600/500{,}000} = 1.26 \text{ km/sec},$$

so that v = 4.19 km/sec.

Problem. 5.27. Establish the formula relating the smallest velocity v_{∞} and the largest velocity v_{π} for hyperbolic motion (Fig. 12).

Solution. Using the formula

$$v_{\infty}^2 = v_{\pi}^2 - \frac{2\mu}{r_{\pi}} \quad \text{and} \quad \mu = \frac{c^2}{p} = \frac{v_{\pi}^2 r_{\pi}^2}{p}$$

let us obtain the relation

$$v_{\infty} = v_{\pi}\sqrt{1 - 2r_{\pi}/p}\,.$$

Knowing that

$$r_{\pi}/p = 1/(e+1)$$

let us determine

$$v_{\infty} = v_{\pi}\sqrt{\frac{e-1}{e+1}}\,, \quad \text{or} \quad v_{min} = v_{max}\sqrt{\frac{e-1}{e+1}} \qquad (5.20)$$

(compare with Eq. (5.10') for elliptical motion).

Problem 5.28. A spacecraft moving along the asymptote of a planetocentric hyperbola approaches a planet with gravitational parameter μ "from infinity" at velocity v_{∞}. The distance of a craft from a pericenter is r_{π}. Determine by what angle 2λ the vector of the planetocentric velocity will rotate after the craft transits the pericenter.

Solution. Since CB = Aπ for a hyperbola (Fig. 12), AO = CO, so that

$$\cos\lambda = \frac{\pi O}{AO} = \frac{|a|}{|a| + r_{\pi}} = \frac{1}{1 + r_{\pi}/|a|}\,,$$

where

$$|a| = \frac{\mu}{v_{\infty}^2}\,.$$

Finally,

$$2\lambda = 2\arccos\left(1 + r_{\pi} v_{\infty}^2/\mu\right)^{-1}.$$

Here the radius-vector of the craft will rotate relative to the launch planet by the angle $\gamma = 180° - 2\lambda$. If, for example, it is not r_π that is known, but the distance of the craft from the planet CB (distance of planet from the asymptote of the hyperbola), by denoting $CB = A\pi = d$, we can write

$$2\lambda = 2\,\mathrm{arctg}\frac{d}{|a|} = 2\,\mathrm{arctg}\frac{v_\infty^2 d}{\mu}\,.$$

Let us illustrate this problem by the following example. Suppose a spacecraft is inserted into an Earth orbit at the altitude $r_{\pi/\oplus} = 230$ km with a hyperbolic launch velocity $v_{\pi/\oplus} = 12.00$ km/sec. The orbit of the craft is situated in the orbital plane /56
of the Moon. The approach of the craft to the Moon at the periselenion is $r_{\pi/\text{☾}} = 6300$ km. In this case, to determine the angles 2λ and γ first from the initial data we must find the elements of the geocentric hyperbola

$$v^2_{\pi/\oplus} = \mu_\oplus\left(\frac{2}{r_{\pi/\oplus}} + \frac{1}{|a|}\right),\quad |a| = 17{,}170 \text{ km},\ e = \frac{r_{\pi/\oplus}}{|a|} + 1 = \frac{6600}{17{,}170} + 1 = 1.384.$$

The theoretical velocity "at infinity" must be determined from the formula

$$v_{\infty/\oplus} = \sqrt{h} = \sqrt{\mu_\oplus/|a|}$$

or from the formula

$$v_{\infty/\oplus} = v_{\pi/\oplus}\sqrt{(e-1)/(e+1)},$$

so that

$$v_{\infty/\oplus} = 4.82 \text{ km/sec}.$$

Since the direction of the asymptote of the geocentric hyperbola coincides at the limit with the direction of the geocentric radius-vector of the craft, it can be assumed that the absolute geocentric velocity of the craft $\bar{v}_{\infty/\oplus}$ is perpendicular to the translational circular velocity of the moon $\bar{v}_{\text{☾}/\oplus}$, which, as was determined in problem 5.5, is 1.02 km/sec. The relative selenocentric velocity of the craft $v_{\infty/\text{☾}}$ is, in accordance with the theorem of the addition of velocities,

$$v_{\infty/\text{☾}} = \sqrt{v^2_{\infty/\oplus} + v^2_{\text{☾}/\oplus}} = \sqrt{(4.82)^2 + (1.02)^2} = 4.92 \text{ km/sec}.$$

Knowing that $\mu_{☾} = 4900$ km^3/sec^2, for the angles 2λ and γ we get

$$2\lambda = 2\arccos\left[1 + \frac{6300(4.92)^2}{4900}\right]^{-1} = 2\arccos 0.033 \approx 176°,$$

$\gamma = 180° - 2\lambda \approx 4°$. The parameters of the selenocentric hyperbola are as follows:

$$|a_{☾}| = \frac{\mu_{☾}}{v^2_{\infty/☾}} = \frac{4900}{(4.92)^2} = 203 \text{ km}, \quad e_{☾} = \frac{r_{\pi/☾}}{|a_{☾}|} + 1 = \frac{6300}{203} + 1 = 30.5,$$

and the selenocentric craft velocity at the periselenion is

$$v_{\pi/☾} = v_{\infty/☾}\sqrt{\frac{e_{☾}+1}{e_{☾}-1}} = 4.92\sqrt{\frac{31.5}{29.5}} = 5.08 \text{ km/sec}.$$

From the velocities obtained $v_{\pi/♁} = 12.00$ km/sec, $v_{\infty/♁} = 4.82$ km/sec, and $v_{\infty/☾} = 4.92$ km/sec, we can determine the minimum angle of rotation of asymptote 2λ that is possible for this problem, corresponding to the case when the hyperbola is tangent to the surface of the Moon ($r_{\pi} = R_{☾} = 1740$ km):

$$2\lambda_{min} = 2\arccos\left(1 + \frac{R_{☾}v^2_{\infty/☾}}{\mu_{☾}}\right)^{-1} = 2\arccos 0.111 \approx 167°.$$

This angle corresponds to the maximum angle of rotation of the geocentric radius-vector $\gamma \approx 13°$.

We can easily show that the values of the angles 2λ and γ lie at the limit

$$2\arccos\left(1 + \frac{Rv^2_{\infty}}{\mu}\right) \leqslant 2\lambda \leqslant 180°, \quad 0 \leqslant \gamma \leqslant 180° - 2\lambda,$$ /57

where R is the radius of the planet. This means that even when tangency is involved the trajectory will not become a hyperbola. So in the case of no values of r_{π} can capture occur for this problem, that is, there is no possible case when the hyperbolic trajectory converts into an elliptical trajectory.

CHAPTER SIX /57

TIME OF MOTION IN A SPACE TRAJECTORY

The time during which a satellite twice successively transits its pericenter is called the period of revolution (orbital period) T of the satellite around the attracting center. The number

$n = \frac{2\pi}{T}$ is its mean angular velocity, or its mean velocity (see problem 2.1). In problem 4.3 the relation of period T and the semi-major axis a with the gravitational parameter of central body μ (4.6) was established:

$$\mu = 4\pi^2 a^3/T^2.$$

The relation between the periods of revolution of different satellites around the same attracting center is characterized by Kepler's third law, established empirically:

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}. \tag{6.1}$$

In the general case of motion in a conic section, the time of motion of a body to an arbitrary point on the trajectory is determined by using the integral of areas $r^2\dot{\phi} = c$ (see problem 3.7), from which in particular we can get the familiar Kepler's equation derived in problem 2.1 from geometrical ratios.

Problem 6.1. Derive formulas for calculating the period of revolution of a "zero" satellite T_{zer} ($r = R_{\delta}$), and also for the period of revolution of an arbitrary Earth satellite T expressed in terms of the period of the "zero" satellite T_{zer}.

Solution. From Kepler's third law (6.1) it follows that

$$\frac{T^2}{T_{zer}^2} = \frac{a^3}{R_\delta^3}, \quad T = T_{zer}\left(\frac{a}{R_\delta}\right)^{3/2}. \tag{6.2}$$

Let us calculate T_{zer} from the formula for uniform circular motion:

$$T_{zer} = \frac{2\pi R_\delta}{v_I} = \frac{2\pi \cdot 6370}{7.91 \cdot 60} = 84.40 \text{ min},$$ /58

so that

$$T = 84.40\left(\frac{a}{6370}\right)^{3/2} \text{min}. \tag{6.3}$$

Problem 6.2. The perigee altitude of a satellite H_π = 230 km, and the apogee altitude H_α = 950 km. What is the period of its revolution?

Solution. By Eq. (6.2) we can write

$$T = T_{zer}\left(\frac{2R_\delta + H_\pi + H_\alpha}{2R_\delta}\right)^{3/2} = T_{zer}\left(1 + \frac{H_\pi + H_\alpha}{2R_\delta}\right)^{3/2} \approx$$
$$\approx T_{zer}\left(1 + \frac{3}{4}\cdot\frac{H_\pi + H_\alpha}{R_\delta}\right) = 84.40\left(1 + \frac{3}{4}\cdot\frac{230 + 950}{6370}\right) = 96.13 \text{ min}.$$

Problem 6.3. Find the period of revolution of a satellite T and its mean motion n if the semi-major axis α and the gravitational parameter of the attracting center μ are known.

Solution. Let us use the result of problem 3.3, in which it was established that when a point moves in an ellipse its radius-vector sweeps out the total ellipse area $\pi \alpha b$ in one period of revolution T, so that $T = 2\pi\alpha b/c$, where c is the constant of areas. Substituting in place of c its value $\sqrt{\mu p}$ from problem 4.3, we get

$$T = \frac{2\pi a b}{\sqrt{\mu p}} = \frac{2\pi a b}{\sqrt{\mu}\sqrt{b^2/a}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}. \tag{6.4}$$

Since $n = \frac{2\pi}{T}$, the mean motion is

$$n = \sqrt{\mu}\, a^{-3/2}. \tag{6.4'}$$

Using Eq. (6.4), for example, we can determine the radius of the circular orbit of a 24-hour AES:

$$a = r = \sqrt[3]{\frac{\mu_{\ast} T^2}{4\pi^2}} = \sqrt[3]{\frac{398,600\,(24\cdot 3600)^2}{4\,(3.14)^2}} = \sqrt[3]{75.5\cdot 10^{12}} = 42,170 \text{ km}.$$

The altitude of the orbit of a 24-hour AES is 35,800 km (see also problem 1.7). /59

Problem 6.4. Show that the period of revolution of a satellite T depends only on its initial velocity and does not depend on its direction.

Solution. Based on Eq. (6.4) we can conclude that the period T is uniquely determined only by the magnitude of the orbital semi-axis. In turn, the semi-major axis α is associated with the algebraic value of the initial velocity v_0 by the integral of energies $v_0^2 = \mu\left(\frac{2}{r_0} - \frac{1}{a}\right)$, so that the period T does not depend on the direction of the initial velocity. This means that if at some point in space a launch is made with elliptical velocity v_0 ($v_{1.ci} < v_0 < v_{1.par}$), for any direction of this velocity ensuring the existence of the flyby trajectory, the satellite will be inserted into orbit with the same semi-major axes α and, therefore, with the same period. These orbits can have different eccentricities. In any case the satellite will return to its launch point after the same time interval.

Problem 6.5. Find the relation between the periods of revolution of planets around the Sun T_i and the semi-major axes of their elliptical trajectories α_i (give the mathematical substantiation of Kepler's third law).

Solution. To justify Kepler's third law let us utilize the familiar relation, which is of the form

$$\frac{\mu_\odot}{4\pi^2} = \frac{a^3}{T^2}.$$

for heliocentric motion. Since the constant $\mu_\odot = 1327\cdot 10^8$ km^3/sec^2, characterizing the Sun's gravitational field appears in the left-hand part of the equation, for all heliocentric orbits the following condition is satisfied:

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2} = \ldots = \frac{a_i^3}{T_i^2} = \frac{\mu_\odot}{4\pi^2}, \qquad (6.5)$$

or

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}, \ldots, \frac{T_1^2}{T_i^2} = \frac{a_1^3}{a_i^3}, \quad (6.5')$$

expressing Kepler's third law. /60

Similarly, for geocentric orbits we can write

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2} = \ldots = \frac{a_i^3}{T_i^2} = \frac{\mu_\delta}{4\pi^2} \qquad \left(\mu_\delta = 398{,}600\ \frac{\text{km}^3}{\text{sec}^2}\right).$$

Problem 6.6. The orbital perihelion of the first Soviet space rocket that became a satellite of the Sun is at a distance from it $r_\rho = 146.4 \cdot 10^6$ km, and its aphelion is at a distance $r_\alpha = 197.2 \cdot 10^6$ km. Determine the period of revolution T of the artificial planet around the Sun.

Solution I. Find the semi-major axis of the orbit of the artificial planet (AP):

$$a = \frac{1}{2}(r_\pi + r_\alpha) = \frac{1}{2}(146.4 + 197.2) \cdot 10^6 = 171.8 \cdot 10^6 \text{ km}.$$

To determine the period of revolution of the AP let us use Eq. (6.4) since the parameter $\mu_\odot = 1327 \cdot 10^8$ km^3/sec^2 is known:

$$T_{AP} = \frac{2\pi a^{3/2}}{\sqrt{\mu_\odot}} = 2\pi\sqrt{\frac{a^3}{\mu_\odot}} = 6.28\sqrt{\frac{(171.8 \cdot 10^6)^3}{1327 \cdot 10^8}} =$$

$$= 6.28\sqrt{38.21 \cdot 10^{12}} = 38.82 \cdot 10^6 \text{ c} = 449.27 \text{ days}$$

Solution II. Knowing the orbital semi-major axis a of the AP, we can use Kepler's third law (6.5'):

$$\frac{T_{AP}^2}{T_\delta^2} = \frac{a_{AP}^3}{a_\delta^3}, \quad \text{Whence when } T_\delta = 365.26 \text{ days } a_\delta = 149.6 \cdot 10^6 \text{ km}$$

$$T_{AP} = T_\delta\sqrt{\left(\frac{a_{AP}}{a_\delta}\right)^3} = 365.26\sqrt{\left(\frac{171.8}{149.6}\right)^3} =$$

$$= 365.26\sqrt{1.52} = 449.27 \text{ days}.$$

Problem 6.7. Using the integral of areas in the equation of a conic section, set up the general integral formula of the time of motion in the conic section as a function of the true anomaly (polar angle) ϕ.

Solution. Using the integral of areas in the form $r^2\dot{\phi} = r^2 \frac{d\varphi}{dt} = c$ and the equation of a conic section $r = \frac{p}{1+e\cos\varphi}$, we get /61

$$dt = \frac{r^2 d\varphi}{c} = \frac{p^2}{c} \cdot \frac{d\varphi}{(1+e\cos\varphi)^2}.$$

Let us reduce the coefficients p^2/c by means of the relation $c = \sqrt{\mu p}$ (see problem 4.3) to the form $p^{3/2}/\sqrt{\mu}$, and then let us integrate the expression for dt from 0 to ϕ (from the pericenter to the arbitrary points):

$$\tau = t - t_\pi = \frac{p^{3/2}}{\sqrt{\mu}} \int_0^\varphi \frac{d\varphi}{(1+e\cos\varphi)^2}, \tag{6.6}$$

where t_π is the moment of transiting of the pericenter.

Problem 6.8. From the integral formula (6.6) and the relations of elliptical motion, by using the substitution

$$\operatorname{tg}\frac{\varphi}{2} = \sqrt{\frac{1+e}{1-e}} \operatorname{tg}\frac{E}{2}$$

(see problem 2.1), derive Kepler's equation for an elliptical trajectory.

Solution. Based on the formula $p = a(1-e^2)$, the integral formula can be written as:

$$\tau = \frac{a^{3/2}(1-e^2)^{3/2}}{\sqrt{\mu}} \int_0^\varphi \frac{d\varphi}{(1+e\cos\varphi)^2}. \tag{6.6'}$$

Then, by means of the substitution indicated in the condition of the problem, we can convert to the variable of integration E (eccentric anomaly). To do this, we must differentiate the expression obtained by substitution, after which we have

$$\sec^2\frac{\varphi}{2}\, d\varphi = \sqrt{\frac{1+e}{1-e}} \sec^2\frac{E}{2}.$$

Further, using the relation $\sec^2\frac{\varphi}{2} = 1 + tg^2\frac{\varphi}{2} = 1 + \frac{1+e}{1-e}\, tg^2\frac{E}{2}$,

we find the expression

$$d\varphi = \sqrt{\frac{1+e}{1-e}} \cdot \frac{dE}{\cos^2\frac{E}{2} + \frac{1+e}{1-e}\sin^2\frac{E}{2}} .$$

Employing the formulas

$$\cos^2\frac{E}{2} = \frac{1}{2}(1+\cos E), \quad \text{and} \quad \sin^2\frac{E}{2} = \frac{1}{2}(1-\cos E)$$

we can write

$$d\varphi = \sqrt{\frac{1+e}{1-e}} \cdot \frac{(1-e)dE}{1-e\cos E} = \frac{\sqrt{1-e^2}\, dE}{1-e\cos E} .$$

We can also express the integrand by E, by employing the relations of elliptical motion

$$r = a(1-e\cos E) \quad \text{and} \quad r\cos\varphi = a(\cos E - e)$$

(see problem 2.1): $1 + e\cos\varphi = \frac{1-e^2}{1-e\cos E}$.

Finally, the integral becomes

$$\tau = \frac{a^{3/2}(1-e^2)^{3/2}}{\sqrt{\mu}} \int_0 \frac{1-e\cos E}{(1-e^2)^{3/2}} dE = \frac{a^{3/2}}{\sqrt{\mu}}(E - e\sin E). \qquad (6.7)$$

Using the formula $n = \sqrt{\mu}\, a^{-3/2}$, we get Kepler's equation that we already derived in problem 2.1 by another method:

$$E - e\sin E = n(t - t_\pi) = M .$$

This expression served in the determination of the time of motion in an elliptical trajectory. It can also be solved for E for known t. In this case Kepler's equation is solved by the method of iterations.

Remark. We can similarly obtain Kepler's equation for a hyperbolic trajectory by using the analog of the eccentric anomaly H = iE, which has real values. The corresponding formulas when e > 1 are of the form

$$\operatorname{tg}\frac{\varphi}{2}=\sqrt{\frac{e+1}{e-1}}\ \operatorname{th}\frac{H}{2}, \quad p=|a|(e^2-1),$$
$$r\cos\varphi=|a|(e-\operatorname{ch}H), \quad r=|a|(e\operatorname{ch}H-1),$$
$$e\operatorname{sh}H-H=n(t-t_\pi)=\sqrt{\mu}\ |a|^{-3/2}(t-t_\pi).$$

Problem 6.9. A spacecraft is flying to the Moon in an elliptical geocentric trajectory tangent to the lunar orbit at its apogee $\alpha_{☾}$ (Fig. 9). Determine the time of flight of the craft /63 by the apogee of the lunar orbit $\alpha_{☾}$ (orbit I). Solve the problem for the case when the flight is made at the perigee of the lunar orbit $\pi_{☾}$ (orbit II) if the perigee and apogee distances of the Moon are $r_{\pi☾}$ = 363,300 km and $r_{\alpha☾}$ = 405,500 km, respectively.

Solution I. In problem 5.20 we determined the major semi-axes of the transfer orbits I and II, a_I = 206,000 km and a_{II} = = 185,000 km.

Now let us write Kepler's third law (6.5) for geocentric motion:

$$\frac{a_I^3}{T_I^2}=\frac{a_{II}^3}{T_{II}^2}=\frac{\mu_\delta}{4\pi^2}=\frac{398600}{4\,(3.14)^2}=1.011\cdot10^4\ \frac{\text{km}^3}{\text{sec}^2},$$

whence the total periods of revolution in orbits I and II are

$$T_I=\sqrt{\frac{a_I^3}{1.011\cdot10^4}}=\sqrt{\frac{(2.06\cdot10^5)^3}{1.011\cdot10^4}}=\sqrt{8.65\cdot10^{11}}=$$
$$=9.30\cdot10^5\text{sec}=10.76\text{days}=10\text{days}\ 18.2\text{ hr}$$
$$T_{II}=\sqrt{\frac{a_{II}^3}{1.011\cdot10^4}}=\sqrt{\frac{(1.85\cdot10^5)^3}{1.011\cdot10^4}}=\sqrt{6.26\cdot10^{11}}=$$
$$=7.91\cdot10^5\text{sec}\ 9.16\text{days}=9\text{days}\ 03.8\text{hr}$$

The half-periods we seek are, respectively: τ_I = 5.38 days = = 5 days 09.2 hours, τ_{II} = 4.58 days = 4 days 13.9 hours.

Solution II. The quantities τ_I and τ_{II} can be obtained directly from Kepler's equation (6.7), by substituting instead

of E its value E = π:

$$\tau_{I,II} = \frac{a_{I,II}^{3/2}}{\sqrt{\mu_\delta}}(E - e\sin E) = \frac{\pi a_{I,II}^{3/2}}{\sqrt{\mu_\delta}} = \frac{3.14 \cdot a_{I,II}^{3/2}}{\sqrt{398600}} = 0.498 \cdot 10^{-2} a_{I,II}^{3/2},$$

$$\tau_I = 0.498 \cdot 10^{-2}\ a_I^{3/2} = 0.498 \cdot 10^{-2} \cdot 9.40 \cdot 10^7 = 4.66 \cdot 10^5 \text{sec} = 5.38 \text{days} = 5 \text{days}\ 09.2 \text{ hr}$$

$$\tau_{II} = 0.498 \cdot 10^{-2} a_{II}^{3/2} = 0.498 \cdot 10^{-2} \cdot 7.98 \cdot 10^7 = 3.97 \cdot 10^5 \text{sec} = 4.58 \text{days} = 4 \text{days}\ 13.9 \text{hr}$$

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Problem 6.10. A spacecraft moving in an Earth orbit is given an additional velocity impulse at point A bringing it into a transfer heliocentric elliptical orbit I tangent to the orbit of Mars at point B (Fig. 13). Determine the flight time to Mars' orbit. Solve the problem for the case of flight to Venus' orbit (orbit II). The orbits of the planets are assumed to be circular with radii $r = 150 \cdot 10^6$ km, $r_1 = 228 \cdot 10^6$ km, and $r_2 = 108 \cdot 10^6$ km.

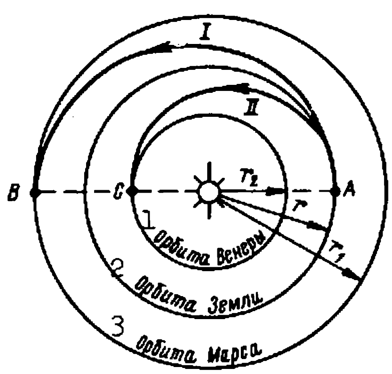

Fig. 13
Key: 1. Venus' orbit
2. Earth's orbit
3. Mars' orbit

Solution. The point of tangency of the orbits B is the aphelion of orbit I, and the point of tangency C is the perihelion of orbit II, since at the points of tangency the vector of the orbital velocity is perpendicular to the radius of the circle of the orbit. Let us determine the semi-major axes of the transfer orbits:

$$a_I = \frac{1}{2}(r + r_1) = 189 \cdot 10^6 \text{ km},$$
$$a_{II} = \frac{1}{2}(r + r_2) = 129 \cdot 10^6 \text{ km}.$$

Knowing that the period of revolution of the Earth around the Sun is 365.26 ≈ 365.3 days, let us use Kepler's third law:

$$T_{I,II} = 365.3\sqrt{\left(\frac{a_{I,II}}{150 \cdot 10^6}\right)^3}, \text{ whence } T_I = 365.3\sqrt{\left(\frac{189}{150}\right)^3} = 515.1 \text{ days},$$
$$T_{II} = 365.3\sqrt{\left(\frac{129}{150}\right)^3} = 291.2 \text{ days},$$

so that the flight time to the orbits of Mars and Venus in an elliptical trajectory is τ_I = 257.6 days and τ_{II} = 146.0 days.

Problem 6.11. A spacecraft will move in a geocentric trajectory transecting the Moon's orbit (Fig. 14). The insertion into orbit occurs at the altitude H = 230 km at a horizontal initial velocity v_0 = 10.95 km/sec. Considering that the Moon will move in a circular orbit with radius $r_{☾}$ = 384,400 km, determine the flight time to the Moon's orbit. /65

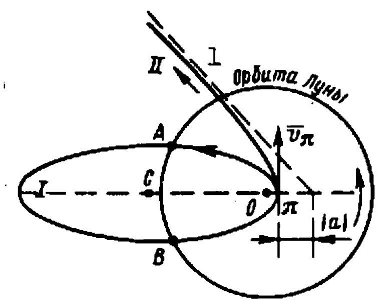

Fig.14
Key: 1. Moon's orbit

Solution. A comparison of the initial launch velocity v_0 = = 10.95 km/sec with a local parabolic velocity at the launch point

$v_{1.par} = \sqrt{\frac{2\mu_\delta}{6600}} =$ 10.99 km/sec

(see also problem 5.20) enables us to conclude that the flight occurs in a strongly elongated elliptical orbit, since the launch velocity does not exceed the parabolic velocity, but slightly differs from it. Let us determine the semi-major axis of the transfer orbit from the integral of energy written for the perigee (the launch is horizontal):

$$v_\pi^2 = \mu_\delta\left(\frac{2}{R_\delta + H_\pi} - \frac{1}{a}\right).$$

Hence from the given $v_0 = v_\pi$ and $H = H_\pi$, we find α = 450,000 km. Let us determine the orbital eccentricity by the formula r_π = = α (1 - e), whence we have

$$e = 1 - r_\pi/a = 1 - \frac{6600}{450{,}000} = 0.985.$$

Thus, the ship executes the flight in a strongly elongated geocentric elliptical orbit I, whose center C, as we can easily determine, lies beyond the Moon's orbit (Fig. 14). The time of motion in the arc of the trajectory πA of interest to us (from the perigee to the point of intersection with the Moon's orbit) can be found from Kepler's equation (6.7): /66

$$\tau = \frac{a^{3/2}}{\sqrt{\mu_\delta}}\,(E - e\sin E).$$

We can determine the eccentric anomaly of the point of intersection E in advance from the equality $r = \alpha\,(1 - 3\cos E) = 384{,}400$ km, whence

$$\cos E = \frac{\alpha - r}{\alpha e} = \frac{450{,}000 - 384{,}400}{450{,}000 \cdot 0.985} = 0.148\,,$$

$$\sin E = 0.989\,, \qquad E = 1.422 = 81^\circ 29'.$$

Substituting these quantities into Kepler's equation, we have

$$\tau = \frac{(450{,}000)^{3/2}}{\sqrt{398{,}600}}\,(1.422 - 0.985 \cdot 0.989) = 2.14 \cdot 10^5 \text{ sec} =$$

$$= 59.4\text{hr} = 2.46 \text{ days.}$$

Remark. We can consider as the point of intersection also the point B, for which when $\cos E_B = 0.146$, $E_B = 278^\circ 24'$, and $\sin E_B = -0.989$, so that

$$\tau = \frac{(450{,}000)^{3/2}}{\sqrt{398{,}600}}\,(1.422 + 0.985 \cdot 0.989) = 11.47 \cdot 10^5 \text{ sec} =$$

$$= 318.6\text{hr} = 13.31 \text{ days.}$$

Problem 6.12. Solve problem 6.11 by assuming that the initial velocity $v_0 = 12.00$ km/sec. /67

Solution. By comparing the given initial velocity with the local parabolic velocity $v_{1.par} = 10.99$ km/sec, let us conclude that since $v_0 > v_{1.par}$, hyperbola II is the flight trajectory (Fig. 14). By writing the integral of energy for the hyperbola (see problem 5.5)

$$v_x^2 = \mu_\delta\,(2/r_x + 1/|\alpha|),$$

let us calculate the semi-major axis:

$$|\alpha| = 17{,}170 \approx 2.7\,R_\delta.$$

From the relation $r_\pi = |a|(e-1)$, we get the eccentricity

$$e = \frac{r_\pi}{|a|} + 1 = \frac{6600}{17{,}170} + 1 = 1.384.$$

To determine the flight time to the Moon's orbit τ, let us use the formulas of hyperbolic motion obtained in problem 6.8:

$$r = |a|\,(e\,\mathrm{ch}\,H - 1),$$
$$e\,\mathrm{sh}\,H - H = \sqrt{\mu_\delta}\,|a|^{-3/2}(t - t_\pi)$$

(Kepler's equation), which in our case can be written as:

$$r_A = |a|\,(e\,\mathrm{ch}\,H - 1) = 384{,}400 \text{ km},$$
$$\tau = t - t_\pi = \sqrt{\frac{|a|^3}{\mu_\delta}}\,(e\,\mathrm{sh}\,H - H).$$

From the first relation we find

$$e\,\mathrm{ch}\,H = \frac{384{,}400}{17{,}170} + 1 = 22.40 + 1 = 23.40,$$
$$\mathrm{ch}\,H = \frac{23.40}{1.384} = 16.91, \quad \mathrm{sh}\,H = \sqrt{\mathrm{sh}^2 H - 1} = 16.88,$$
$$e\,\mathrm{sh}\,H = 23.36, \quad H = 3.52 = 201°41',$$

which, on being substituted into the second, allow us to determine

$$\tau_A = \sqrt{\frac{(17{,}170)^3}{398{,}600}}\,(23.36 - 3.52) = 3.56 \cdot 10^3 \cdot 19.84 =$$
$$= 7.06 \cdot 10^4 \text{ sec} = 19.6 \text{ hr}.$$

Problem 6.13. Derive the approximate formula for the flight time in an elliptical trajectory as a function of true anomaly (polar angle) ϕ for small eccentricities. /68

Solution. Let us use Eq. (6.6') (see problem 6.8) derived for the elliptical trajectories:

$$\tau = \frac{a^{3/2}(1-e^2)^{3/2}}{\sqrt{\mu}} \int_0^{\varphi} \frac{d\varphi}{(1+e\cos\varphi)^2}.$$

Let us expand the integrand in a series in powers of eccentricity e. Limiting ourselves to the first power of e, we get

$$\tau \approx \frac{a^{3/2}}{\sqrt{\mu}} \int_0^{\varphi} (1-2e\cos\varphi)d\varphi = \frac{a^{3/2}}{\sqrt{\mu}}(\varphi - 2e\sin\varphi). \tag{6.8}$$

Since the mean velocity $n = 2\pi/T = \sqrt{\mu}\, a^{-3/2}$, we have

$$\tau \approx \frac{T}{2\pi}(\varphi - 2e\sin\varphi). \tag{6.9}$$

Problem 6.14. A spacecraft satellite inserted into Earth orbit has a perigee altitude H_π = 180 km and an apogee altitude H_α = 340 km. The moment the ship transits the perigee t_π = 9 hr 00 min. In the next revolution it is required to fire the retro-engine at the moment when the true anomaly of the craft ϕ = 270°. Determine the semi-major axis of the orbit of the spacecraft α, its eccentricity e, the period of revolution of the craft T, and the moment of firing the retro-engine t*.

Solution. From the given altitudes H_π and H_α let us find α, r_π, and e:

$$a = R_\delta + \tfrac{1}{2}(H_\pi + H_\alpha) = 6630 \text{ km}, \quad r_\pi = R_\delta + H_\pi = 6550 \text{ km},$$

but $\quad r_\pi = a(1-e), \quad$ whence $\quad e = a - \frac{r_\pi}{a} = \frac{80}{6630} \approx 0.012.$

From Kepler's third law written in the form of Eq. (6.3') (see problem 6.1), let us determine T:

$$T = 84.4\left(\frac{a}{6370}\right)^{3/2} = 84.4\left(\frac{6630}{6370}\right)^{3/2} = 89.5 \text{ min}$$

Considering that the eccentricity (e = 0.012) is quite small, let us assume that the approximate formula of the flight time (6.9) obtained in problem 6.13: /69

$$\tau = t - t_\pi = \frac{T}{2\pi}(\varphi - 2e\sin\varphi),$$

whence we get for the time of flight from the perigee to the instant the retro-engine is fired ($\phi = 3/2\ \pi$):

$$\tau\Big|_{\varphi=270^\circ} = \frac{89{,}5}{2\pi}\left(\frac{3}{2}\pi + 0{,}024\right) = 66{,}7 \text{ min.}$$

Let us determine the moment of firing the retro-engine:

$$t^* = t_\pi + \tau = 9 \text{ hr } 00 \text{ min} + 66.7 \text{ min} = 10 \text{ hr } 06.7 \text{ min.}$$

Problem 6.15. From the integral formula (6.6) (see problem 6.8) and the relations of parabolic motion, derive the formula of flight time in a parabolic trajectory from the pericenter π to the point A with given polar coordinates r_A and ϕ_A (Fig. 11).

Solution. For a parabola e = 1, therefore the integral formula (6.6) can be rewritten in the simplified form:

$$\tau = \frac{p^{3/2}}{\sqrt{\mu}}\int_0^{\varphi_A}\frac{d\varphi}{(1+\cos\varphi)^2} = \frac{p^{3/2}}{4\sqrt{\mu}}\int_0^{\varphi_A}\frac{d\varphi}{\cos^4\frac{\varphi}{2}} .$$

Using the relation

$$1/\cos^2\frac{\varphi}{2} = 1 + \operatorname{tg}^2\frac{\varphi}{2},$$

let us write

$$\tau(\varphi_A) = \frac{p^{3/2}}{2\sqrt{\mu}}\int_0^{\operatorname{tg}\frac{\varphi_A}{2}}\left(1+\operatorname{tg}^2\frac{\varphi}{2}\right)d\left(\operatorname{tg}\frac{\varphi}{2}\right) = \frac{p^{3/2}}{2\sqrt{\mu}}\operatorname{tg}\frac{\varphi_A}{2}\left(1+\frac{1}{3}\operatorname{tg}^2\frac{\varphi_A}{2}\right).$$

Thus, τ can easily be expressed not only as a function of ϕ_A, but also as a function of r_A, for which let us use the equation of the

$$r = \frac{p}{1+\cos\varphi} = \frac{p}{2}\left(1+\operatorname{tg}^2\frac{\varphi}{2}\right),$$

whence

$$\operatorname{tg}\frac{\varphi}{2} = \sqrt{\frac{2r}{p}-1}\ , \quad 1 + 3\operatorname{tg}^2\frac{\varphi}{2} = \frac{2}{3}\left(\frac{r}{p}+1\right).$$

We finally get

$$\tau(r_A) = \frac{p^{3/2}}{3\sqrt{\mu}}\sqrt{\frac{2r_A}{p}-1}\left(\frac{r_A}{p}+1\right) = \frac{\sqrt{2r_A-p}\,(r_A+p)}{3\sqrt{\mu}}. \tag{6.10}$$

In this formula parameter p can be replaced by its value $p = 2r_\pi$, characterizing the parabolic trajectory, so that

$$\tau(r_A) = \frac{1}{3\sqrt{\mu}}\sqrt{2(r_A - r_\pi)}\ (r_A + 2r_\pi). \tag{6.10'}$$

Problem 6.16. A rocket at an altitude H_π = 230 km above the Earth's surface is given a horizontal parabolic velocity. Determine the flight time over a distance equal to the mean radius of the Moon's orbit $r_☾$ = 384,400 km.

Solution. Let us use Eq. (6.10'), by setting $r_A = r_☾ =$ = 384,400 km, $r_\pi = R_♁ + H_\pi$ = 6600 km:

$$\tau = \frac{\sqrt{2(384400-6600)}}{3\sqrt{398600}}(384400+13200) = 1.82\cdot 10^5 \text{ sec} = 50.6 \text{ hr}.$$

It is of interest to compare this result with the results of problem 6.11 and 6.12: flight in an elliptical trajectory takes 59.4 hours, while it takes 19.6 hours in a hyperbolic trajectory. The solution for the problem of flight in the parabolic trajectory is simpler, since it does not require the integral of energy. For this reason, we need not use the parabolic velocity in calculations. The local parabolic velocity for the altitude H = 230 km was already determined in problem 5.20 (v = 10.99 km/sec).

CHAPTER SEVEN /70

CONDITIONS FOR THE EXISTENCE OF ELLIPTICAL TRAJECTORIES

Among the elliptical trajectories, in which the main conditions for existence is the velocity constraints

$$v_{1.ci} < v < v_{1.par}, \tag{7.1}$$

we will distinguish two classes of trajectories -- flyby and ballistic. For a body launched from the Earth's surface ($r_0 = R_♁$) to describe about it a closed elliptical curve ($0 < e < 1$), that is, to enter a flyby trajectory, the flyby condition

$$r > r_0 = R_♁ \tag{7.2}$$

must be satisfied (the case of a "zero" satellite when $r = r_0 = R_♁$, $v = v_I$, and $e = 0$ is not considered by us here, although it also can be included in the class of flyby trajectories). For a launch from altitude H above the Earth's surface these conditions (the circular orbit is included) become

$$v_{1.ci} \leqslant v < v_{1.par}, \tag{7.1'}$$

$$r \geqslant r_0. \tag{7.2'}$$

In addition to these obvious and necessary flyby conditions, there are yet a number of other conditions that will be established in the following problems.

An elliptical trajectory which often occurs within the Earth ($r < R_♁$) and motion along it is possible only outside it, that is,

where $r \geqslant R_\delta$, is called a ballistic trajectory. Below we will /71
establish the criteria that enable us to determine whether a trajectory belongs to a particular class of elliptical trajectories.

Problem 7.1. Show that for a flyby to be feasible, in addition to satisfying condition (7.1) and (7.2), it is necessary that the launch point at the Earth's surface be the perigee of the orbit and the velocity at this point be directed horizontally.

Solution. This condition follows directly from condition (7.2), since if the launch point is at the surface of the Earth, for a flyby trajectory tangent to the Earth at the launch point this point can be nowhere else except the perigee. In Section 3.5 it was shown that the orbital velocity at the perigee and apogee is perpendicular to the radius-vector, so that the launch velocity in this case must be horizontal.

However, this statement can be proven more rigorously. Actually, the equation of a conic section $p/(1+e\cos\varphi) > p/(1+e\cos\varphi_0)$ corresponds to the flyby condition (7.2). When ϕ is varied from 0 to π this conditions means that $\phi > \phi_0$. The latter is possible only when $\phi = 0$ (the launch point coincides with the perigee).

Now let us show that when the launch point coincides with the perigee $\alpha_0 = \pm\frac{\pi}{2}$, $\beta_0 = 0, \pi$ (Fig. 15), where $\alpha_0 = (\bar{r}_0 \,\hat{,}\, \bar{v}_0)$, and $\beta_0 = \frac{\pi}{2} - \alpha_0$ is the angle of $\bar{v}_0$ with the local horizon. Let us use Eqs. (4.14) and (4.15) (see problem 4.10) written for $0 \leqslant \varphi \leqslant \pi$:

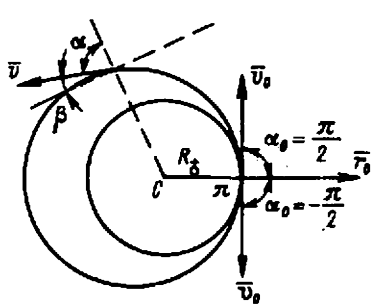

Fig. 15

$$e\sin\varphi_0 = \frac{c^2}{\mu_\delta}\sqrt{\frac{v_0^2}{c^2} - u_0^2},$$

$$e\cos\varphi_0 = \frac{u_0 c^2 - \mu_\delta}{\mu_\delta}.$$

Knowing that $\mu_\delta = gR_\delta^2$ (see problem 4.7) and $c = |\bar{R}_\delta \times \bar{v}_0| = R_\delta v_0 \sin\alpha$ (see problem 3.4), we can write /72

$$e \sin \varphi_0 = \frac{R_\delta^2 \; v_0^2 \; \sin^2\alpha_0}{gR_\delta^2} \sqrt{\frac{v_0^2}{R_\delta^2 \; v_0^2 \; \sin\alpha_0} - \frac{1}{R_\delta^2}} = \frac{v_0^2 \; \sin^3\alpha_0 \; \mathrm{ctg}\, \alpha_0}{gR_\delta},$$

$$e \cos \varphi_0 = \frac{R_\delta^2 \; v_0^2 \; \sin^2\alpha_0}{gR_\delta^3} - 1 = \frac{v_0^2 \; \sin^2\alpha_0}{gR_\delta} - 1.$$

When $\phi_0 = 0$, Eqs. (7.3) can be reduced to the following expressions:

$$\sin^3\alpha_0 \, \mathrm{ctg}\, \alpha_0 = 0, \quad e = \frac{v_0^2 \; \sin^2\alpha_0}{gR_\delta} - 1. \qquad (7.4)$$

For both equations (7.4) to be satisfied, ctg α_0 must be equal to zero, since sin α_0 cannot tend to zero by virtue of the condition $0 < e < 1$. Therefore, $\alpha = \pm \pi/2$, and $\beta_0 = 0, \pi$, and this means that the launch is horizontal.

Thus, for a flyby to be possible from a launch from the Earth's surface simultaneously three conditions must be satisfied:

$$v_{1.ci} < v_0 < v_{1.par}; \; \varphi_0 = 0; \; \alpha_0 = \pm\frac{\pi}{2}, \; \beta_0 = 0, \pi. \qquad (7.5)$$

If $\alpha \neq \pi/2$, for any initial velocity the body launched from the Earth's surface cannot be inserted into flyby trajectory. In this case only a ballistic trajectory can obtain.

When a body is launched from altitude H above the Earth's surface, satisfying condition (7.5) as before ensures a flyby, however they are not mandatory for its execution. This means that the insertion into orbit can be made at an arbitrary point of the orbit, which is neither at the perigee or the apogee. The latter fact presupposes violation of the condition that the velocity $\bar{v}_0$ is horizontal.

Problem 7.2. A satellite is inserted into elliptical orbit at altitude H above the Earth's surface with velocity $\bar{v}_0$ directed horizontally (along the tangent to the circle drawn through the launch point from the Earth's center). For the satellite to move around the Earth for an extended time it must not approach its surface more closely than the distance h(h < H). What must the launch velocity v_0 be to ensure this condition? /73

Solution I. From the condition of the problem it follows that if the launch velocity is directed horizontally, the launch point at altitude H can only be the orbital apogee

$$(r_{max} = r_\alpha = R_\delta + H),$$

since the satellite can attain a lower altitude (h < H). The constraing h < H means that the perigee distance r_π must obey the condition

$$r_{min} = r_\pi \geqslant R_\delta + h.$$

Let us write the formula of the launch velocity $v_0 = v_\alpha$ analogously to (5.16) (see problem 5.19):

$$v_\alpha^2 = \frac{\mu_\delta}{a} \cdot \frac{r_\pi}{r_\alpha} = \frac{2\mu_\delta}{r_\alpha} \cdot \frac{r_\pi}{2a} = \frac{2\mu_\delta}{R_\delta + H} \times$$

$$\times \frac{r_\pi}{R_\delta + H + r_\pi} = \frac{2\mu_\delta}{R_\delta + H}\left(1 - \frac{R_\delta + H}{R_\delta + H + r_\pi}\right).$$

Let us evaluate the expression in the parentheses, beginning with the constraints $r_\pi \geqslant R_\delta + h,$ and $R_\delta + H + r_\pi \geqslant 2R_\delta + H + h:$

$$\frac{R_\delta + H}{R_\delta + H + r_\pi} \leqslant \frac{R_\delta + H}{2R_\delta + H + h}.$$

Therefore,

$$v_\alpha^2 \geqslant \frac{2\mu_\delta}{R_\delta + H}\left(1 - \frac{R_\delta + H}{2R_\delta + H + h}\right)$$

and

$$v_0 = v_\alpha \geqslant \sqrt{\frac{2\mu_\delta}{2R_\delta + H + h} \cdot \frac{R_\delta + h}{R_\delta + H}}.$$

The problem always has a solution, since the radicand is always positive.

Solution II. The problem does have a simpler solution.

Let us write the integral of energy for the apogee and the perigee:

$$v_\alpha^2 - \frac{2\mu_\text{з}}{R_\text{з}+H} = v_\pi^2 - \frac{2\mu_\text{з}}{R_\text{з}+h},$$

whence /74

$$v_\alpha^2 = v_\pi^2 - 2\mu_\text{з}\frac{H-h}{(R_\text{з}+H)(R_\text{з}+h)}.$$

Let us replace v_π by its value $v_\pi = \frac{R_\text{з}+H}{R_\text{з}+h} v_\alpha$, deriving from the integral of areas, after which we get

$$v_\alpha = \sqrt{\frac{2\mu_\text{з}(H-h)(R_\text{з}+h)}{(R_\text{з}+H)\left[(R_\text{з}+H)^2-(R_\text{з}+h)^2\right]}} = \sqrt{2\mu_\text{з}\frac{R_\text{з}+h}{R_\text{з}+H}\cdot\frac{1}{2R_\text{з}+H+h}}.$$

This then is the expression of velocity $v_0 = v_\alpha$ required for the perigee altitude to exactly equal h. In order for this altitude not to be less than the given value h, it is necessary that

$$v_0 = v_\alpha \geqslant \sqrt{2\mu_\text{з}\frac{R_\text{з}+h}{R_\text{з}+H}\cdot\frac{1}{2R_\text{з}+H+h}}. \qquad (7.6)$$

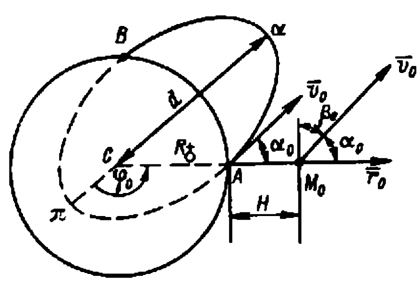

Fig. 16

Problem 7.3. A rocket is launched at an arbitrary angle α_0 to the vertical at the Earth's surface. Find the minimum initial launch velocity v_0 required for the maximum separation of the rocket from the Earth's surface r_α to be equal to the assigned distance d (Fig. 16). Determine the semi-major axis a and the eccentricity e of the corresponding elliptical orbit as a function of angle α_0.

Solution. As follows from the solution to problem 7.1, the trajectory cannot be a flyby trajectory since when the rocket is launched from the Earth's surface at point A the flyby is feasible only in the particular case when $\alpha = \pm\pi/2$. In this case motion occurs in a ballistic trajectory, more accurately, along the arc AαB, (the arc BπA lies within the Earth).

Let us write the integral of energy for the launch point A and the apogee α:

$$v_0^2 - \frac{2\mu_\delta}{R_\delta} = v_\alpha^2 - \frac{2\mu_\delta}{d} .$$

Simultaneously, from the integral of energies we have /75

$$c = |\bar{R}_\delta \times \bar{v}_0| = |\bar{r}_\alpha \times \bar{v}_\alpha| = R_\delta\, v_0 \sin\alpha_0 = d\, v_\alpha ,$$

$$v_\alpha = \frac{R_\delta}{d}\, v_0 \sin\alpha_0 .$$

Substituting the value v_α we have obtained into the integral of energy, we get

$$v_0^2 \left(1 - \frac{R_\delta^2}{d^2} \sin^2\alpha_0\right) = 2\mu_\delta\left(\frac{1}{R_\delta} - \frac{1}{d}\right),$$

so that the desired launch velocity is

$$v_0 = \sqrt{\frac{2\mu_\delta\, d\,(d - R_\delta)}{R_\delta\,(d^2 - R_\delta^2 \sin^2\alpha_0)}}. \qquad (7.7)$$

Let us determine the semi-major axis of the ballistic trajectory α from the integral of energy $v_0^2 = \mu_\delta\left(\frac{2}{R_\delta} - \frac{1}{a}\right)$, so that

$$a = \frac{\mu_\delta R_\delta}{2\mu_\delta - v_0^2 R_\delta} = \frac{d^2 - R_\delta^2 \sin^2\alpha_0}{2(d - R_\delta \sin^2\alpha_0)}. \qquad (7.7')$$

We can find the eccentricity from the apogee distance $r_\alpha = d = a(1+e)$, from whence

$$e = \frac{d - a}{a} = \frac{d}{a} - 1 = \frac{2d\,(d - R_\delta \sin^2\alpha_0)}{d^2 - R_\delta^2 \sin^2\alpha_0} - 1 = \frac{d - R_\delta \sin^2\alpha_0 (2d - R_\delta)}{d^2 - R_\delta^2 \sin^2\alpha_0}.$$

We can easily see that from the resulting formulas for $\alpha = \pm\, \pi/2$, the formulas we already obtained in problem 5.19 for the flyby trajectory follow as a particular case:

$$a = \frac{1}{2}(d + R_\delta) = \frac{1}{2}(r_\alpha + r_\pi),$$

$$v_0^2 = v_\pi^2 = \frac{2\mu_\delta d}{R_\delta(d + R_\delta)} = \frac{2\mu_\delta r_\alpha}{r_\pi(r_\alpha + r_\pi)} = \frac{\mu_\delta}{a} \cdot \frac{r_\alpha}{r_\pi}.$$

Problem 7.4. A material point moving according to the law of universal gravity is at the position M_0 at the initial moment (Fig. 16), at the distance $r_0 = R_\delta + H$ from the Earth's center and has the velocity $\bar{v}_0$. The angle between the velocity center $\bar{v}_0$ and the line of the local horizon is β_0, and the polar angle of the point M_0 (true anomaly) is ϕ_0. Determine the eccentricity of the elliptical orbit and establish the dependence of angles ϕ_0 and β_0. Is it possible from these problem conditions to determine whether the trajectory is a flyby or ballistic trajectory? /76

Solution. To find the eccentricity, let us use the general formula

$$e = \sqrt{1 + \frac{c^2}{\mu_\delta^2}\tilde{h}}, \tag{7.8}$$

where

$$c = r_0 v_0 \sin(\widehat{\bar{r}_0, \bar{v}_0}) = r_0 v_0 \cos\beta_0, \quad \tilde{h} = v_0^2 - \frac{2\mu_\delta}{r_0}, \tag{7.8'}$$

so that from the assigned r_0, v_0, and β_0 we can uniquely determine c, $\tilde{h}$, and e.

To establish the relation between the angles ϕ_0 and β_0, let us use relations (7.3), writing them in our case as

$$e \sin\varphi_0 = \frac{r_0^2 v_0^2 \cos^2\beta_0}{g_H r_0^2}\sqrt{\frac{v_0^2}{r_0^2 v_0^2 \cos^2\beta_0} - \frac{1}{r_0^2}} = \frac{v_0^2 \cos^2\beta_0 \operatorname{tg}\beta_0}{g_H r_0},$$

$$e \cos\varphi_0 = \frac{r_0^2 v_0^2 \cos^2\beta_0}{g_H r_0^3} - 1 = \frac{v_0^2 \cos^2\beta_0}{g_H r_0} - 1,$$

where g_H is the acceleration due to gravity at altitude H (see problem 1.6). Dividing the first equation by the second, we get

$$\operatorname{tg}\varphi_0 = \frac{v_0^2 \cos^2\beta_0 \operatorname{tg}\beta_0}{v_0^2 \cos^2\beta_0 - g_H r_0} = \frac{\operatorname{tg}\beta_0}{1 - g_H r_0 / v_0^2 \cos^2\beta_0}.$$

Transforming the denominator of the resulting fraction by means of the relations

$$v_0^2 \cos^2\beta_0 = \frac{c^2}{r^2}, \quad \mu_\delta = \frac{c^2}{p}, \quad g_H r_0 = v_{1ci}^2 = \frac{\mu_\delta}{r_0} = \frac{c^2}{p r_0}, \tag{7.9}$$ /77

we find

$$\operatorname{tg}\varphi_0 = \frac{\operatorname{tg}\beta_0}{1 - \frac{r_0}{p}} = \frac{\operatorname{tg}\beta_0}{1 - \frac{r_0}{a(1-e^2)}}. \tag{7.10}$$

The orbital parameter $p - c^2/\mu_\delta$ can be assumed known since from (7.8') we know the constant of areas. Since the eccentricity then can be assumed known, then consequently we also know the semi-axis α. If angle ϕ_0 had not been given in the conditions, Eq. (7.10) could serve for its determination from the known β_0, r_0, α, and e.

We can at once from the problem conditions establish whether the trajectory is a flyby or ballistic trajectory. To do this, it is sufficient to find the perogee distance $r_{min} = r_\pi = \alpha\ (1 - e)$ and to compare it with the Earth's radius, by verifying condition (7.1).

Problem 7.5. At which point of the ballistic trajectory does the angle of trajectory inclination to the local axis (plane perpendicular to the radius-vector) reach its largest value?

Solution. Let us use the results of problem 7.4. Let M_0 (Fig. 16) be an arbitrary point on an elliptical orbit. We will denote angle β_0 by β, bearing in mind the arbitrary point on the orbit. Eq. (7.10) is valid for any point on the orbit:

$$\operatorname{tg}\beta = \operatorname{tg}\varphi\left(1 - \frac{r}{p}\right) = \frac{e \sin\varphi}{1 + e\cos\varphi}.$$

Let us set up derivative $\frac{d\beta}{d\varphi}$ and equate it equal to zero:

$$\frac{d\beta}{d\varphi} = \frac{1}{1+\left(\frac{e \sin\varphi}{1+e\cos\varphi}\right)^2} = \frac{-e^2 - e\cos\varphi}{(1+e\cos\varphi)^2} = 0,$$

whence we get $\cos\phi = -e$ for $e \neq 0$. Corresponding to this condition is the radius-vector $r = p/1 + e\cos\varphi = \alpha(1-e^2)/1-e^2 = \alpha$.

However, in problem 5.11 it was shown that the conditions $\cos\phi =$ $= -e$ and $r = \alpha$ are satisfied only for two points on the ellipse B and B' situated at the end points of the semi-minor axis (see Fig. 7), that is, when $E = \pm\pi/2$.

Problem 7.6. At the moment the spacecraft separates from the last rocket stage, it is at point M_0 (Fig. 16) situated at altitude H = 200 km from the Earth's surface, and has an initial velocity v_0 = 8.50 km/sec, where the vector $\bar{v}_0$ makes an angle of $\beta_0 = 10°.00'$ with the line of the local horizon. Calculate the constant of areas with the craft trajectory c, constant of energy $\tilde{h}$, eccentricity e, trajectory parameter p, polar angle of launch point ϕ_0, semi-major axis α, and perigee and apogee distances r_π and r_α. Determine from the data obtained whether the trajectory is a flyby or ballistic trajectory. /78

Solution. Let us use formulas from problem 7.4 for the calculations.

Let us determine the constant of areas by the formula

$$c = r_0 v_0 \cos\beta_0 = (R_\delta + H) v_0 \cos\beta_0 = (6370+200)\cdot 8.50 \cdot 0.985 = 55{,}000 \text{ km}^2/\text{sec}.$$

We calculate the constant of energy $\tilde{h}$ by the formula

$$\tilde{h} = v_0^2 - \frac{2\mu_\delta}{r_0} = (8.50)^2 - \frac{2\cdot 398{,}600}{6570} = 72.25 - 121.34 = -49.1 \ \frac{\text{km}^2}{\text{sec}^2}.$$

We find the eccentricity from (7.8)

$$e = \sqrt{1+\frac{c^2}{\mu_\delta^2}\tilde{h}} = \sqrt{1-49.1\left(\frac{55{,}000}{398{,}600}\right)^2} = \sqrt{1-0.931} = 0.263.$$

Parameter p, according to (7.9) is

$$p = \frac{c^2}{\mu_\delta} = \frac{55{,}000^2}{398{,}600} = 7590 \text{ km}.$$

Determine the polar angle by Eq. (7.10):

$$\operatorname{tg} \varphi_0 = \frac{\operatorname{tg} \beta_0}{1 - \frac{r_0}{p}} = \frac{0{,}176}{1 - \frac{6570}{7590}} = \frac{0{,}176}{0{,}134} = 1{,}310, \ \varphi = 52°38',$$

where we can verify ϕ_0 by using Eq. (4.17):

$$\operatorname{tg} \varphi_0 = \frac{c\sqrt{v_0^2 - \left(\frac{c}{r_0}\right)^2}}{\frac{c^2}{r_0} - \mu_\delta} = \frac{55{,}000\sqrt{8{,}50^2 - \left(\frac{55{,}000}{6570}\right)^2}}{\frac{(55{,}000)^2}{6570} - 398{,}600} = 1{,}310.$$

Let us compute the semi-major axis in terms of parameter p:

$$a = \frac{p}{1 - e^2} = \frac{7590}{1 - 0{,}069} = \frac{7590}{0{,}931} = 8150 \text{ km},$$ /79

and let us find the distances r_π and r_α from a and e:

$$r_\pi = a(1 - e) = 8150(1 - 0{,}263) = 6000 \text{ km},$$
$$r_\alpha = a(1 + e) = 8150(1 + 0{,}263) = 10{,}300 \text{ km}.$$

Since the r_π proves to be smaller than $R_\oplus$, obviously the ballistic is a trajectory one. The flyby is impossible for the given initial conditions.

Problem 7.7. A rocket at a distance r_0 from the Earth's center is given ($r_0 \gtrless R_\oplus$) some initial velocity $\bar{v}_0$ directed at an angle β_0 to the horizon. What must this velocity be for the radius-vector of the perigee to be equal to the given value r_π ($r_\pi < r_0$)?

Solution. As follows from the problem condition, the launch point M_0 (Fig. 16) is neither the perigee nor the apogee of the

orbit. Let us write the integral of areas for the launch point M_0 and the perigee π:

$$c = |\bar{r} \times \bar{v}| = r_0 v_0 \cos\beta_0 = r_\pi v_\pi .$$

Here

$$v_\pi = \frac{r_0 v_0 \cos\beta_0}{r_\pi} .$$

Writing the integral of areas

$$v_0^2 - 2\mu_\oplus / r_0 = v_\pi^2 - 2\mu_\oplus / r_\pi ,$$

for these same two points, in it we substitute the value of v_π:

$$v_0^2 = v_\pi^2 - 2\mu_\oplus \left(\frac{1}{r_\pi} - \frac{1}{r_0}\right) = v_\pi^2 - 2\mu_\oplus \frac{r_0 - r_\pi}{r_0 r_\pi} =$$

$$= \frac{r_0^2 v_0^2 \cos^2\beta_0}{r_\pi^2} - 2\mu_\oplus \frac{r_0 - r_\pi}{r_0 r_\pi} ,$$

whence

$$v_0^2 \left[1 - \left(\frac{r_0}{r_\pi}\right)^2 \cos^2\beta_0\right] = -2\mu_\oplus \frac{r_0 - r_\pi}{r_0 r_\pi}$$

and

$$v_0 = \sqrt{\frac{2\mu_\oplus (r_0 - r_\pi)}{r_0 r_\pi \left[\left(\frac{r_0}{r_\pi}\right)^2 \cos^2\beta_0 - 1\right]}} > 0 . \qquad (7.11)$$

For the problem to have a solution, the inequality $\frac{r_0}{r_\pi} \cos\beta_0 > 1$, /80
must be satisfied, whence

$$\cos\beta_0 > \frac{r_\pi}{r_0} , \qquad (7.12)$$

so that we can determine from the given β_0, r_π, and r_0 whether the assigned distance r_π has been attained.

Eqs. (7.11) and (7.12) enable us to investigate both classes of elliptical trajectories for which $r_\pi \geqslant R_\oplus$ (flyby) or $r_\pi < R_\oplus$

(ballistic). Upon inspection of the flyby trajectories we can assign not r_0 and r_π, but the launch altitude and the perigee altitude H and h_π. In this case (7.11) becomes

$$v_0 = \sqrt{\frac{2\mu_\delta (H - h_\pi)}{(R_\delta + H)(R_\delta + h_\pi)\left[\left(\frac{R_\delta + H}{R_\delta + h_\pi}\right)^2 \cos^2\beta_0 - 1\right]}} \qquad (7.13)$$

where $\cos\beta_0 > (R_\delta + h_\pi)/(R_\delta + H)$, so that actually we determine what velocity must be imparted to the rocket at altitude H for the altitude of its orbital perigee to be equal to the given altitude h_π.

We can easily see that from (7.13) as a particular example Eq. (7.6) follows, obtained for the velocity in horizontal launch. Actually, when $\beta_0 = 0$ we have again

$$v_0 = v_\alpha = \sqrt{\frac{2\mu_\delta (H - h_\pi)}{(R_\delta + H)[(R_\delta + H)^2 - (R_\delta + h_\pi)^2]}} = \sqrt{\frac{2\mu_\delta}{2R_\delta + H + h_\pi}\left(\frac{R_\delta + h_\pi}{R_\delta + H}\right)}.$$

Problem 7.8. Under which initial conditions will the trajectory of a spacecraft launched at altitude H from the surface of a planet with radius R not intersect its surface?

Solution. We can consider this problem as a particular problem of the preceding one, with $r_\pi = R$ ($h_\pi = 0$) (the tangency is included in the case of flyby). By Eq. (7.11) we could easily establish the limiting velocity at which intersection will not occur, that is, in which flyby is observed:

$$v_{lim} = \sqrt{\frac{2\mu (r_0 - R)}{r_0 R\left[\left(\frac{r_0}{R}\right)^2 \cos^2\beta_0 - 1\right]}} = \sqrt{\frac{2\mu R H}{(R+H)[(R+H)^2\cos^2\beta_0 - R^2]}}, \qquad (7.13')$$

so that when $v_0 \geq v_{lim}$, we get the flyby trajectory (the case $v_0 = v_{lim}$ gives the case of tangency). This condition can be represented as: /81

$$v_0 \geq v_{1.par}\sqrt{\frac{RH}{(R+H)^2\cos^2\beta_0 - R^2}} = v_{1.ci}\sqrt{\frac{2RH}{(R+H)^2\cos^2\beta_0 - R^2}}, \qquad (7.13'')$$

where $v_{1.ci} = \sqrt{\frac{\mu}{R+H}}$ and $v_{1.par} = \sqrt{2}\, v_{1.ci}$ are the local circular and local parabolic velocities at the launch point.

Now let us look at the conditions imposed on angle β_0 (with local horizon) or angle α_0 (with the local vertical) (Fig. 16). Obviously, the problem of determining velocity has a solution if the condition $(R+H)^2\cos^2\beta_0 - R^2 > 0,$ is observed, where $(R+H)\cos\beta_0 > R,$ or

$$(R+H)\sin\alpha_0 > R. \tag{7.14}$$

From the geometrical point of view, this condition means that the velocity vector $\bar{v}_0$ must be directed outside a cone described around a planet and with its vertex at the launch point M_0 (Fig. 17), so that the angle of the semi-aperture of the cone $\theta = 180° - \alpha_0$ is a function of the launch point altitude above the surface of the planet H. Actually, from the right triangle M_0CA we have

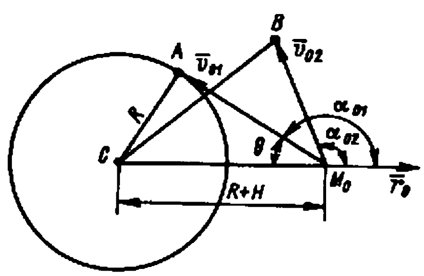

Fig. 17

$$R = (R+H)\sin(180° - \alpha_{01}) = (R+H)\sin\alpha_{01} = (R+H)\cos\beta_{01} \tag{7.15}$$

therefore, if the velocity vector $\bar{v}_{01}$ is directed along the cone generatrix, the problem has no solution. For any other vector $\bar{v}_{02}$ directed outside the cone, from the relation of the sides of ΔM_0CB it is clear that the leg CB = (R + H) sin α_{02} = (R + H) cos β_{02} is larger than the radius of planet R, so that the condition is satisfied. Condition (7.14) and (7.15) are identical to condition (7.12). /82

CHAPTER EIGHT

/82

TRANSFER FROM ORBIT TO ORBIT

If at a point with orbital velocity $\bar{v}$ an addition velocity impulse $\bar{u}$ of arbitrary direction is applied, the point acquires the velocity $\bar{v} + \bar{u}$ and passes into a new orbit. Here the impulse $\bar{u}$ must be specified in magnitude and direction. We will relate the direction of the radial impulse $\bar{u}$ with the positive direction of the unit radius-vector $\bar{r}^{0}$ and assume that $\bar{u} = u\bar{r}^{0}$. Here we will regard the impulse $\bar{u}$ as positive if $u > 0$, and negative if $u < 0$. We will associate the tangential impulse $\bar{u}$ with the positive direction of the tangent to the orbit $\bar{\tau}^{0}$ (directed toward the side of motion) and we will assume that $\bar{u} = u\bar{\tau}^{0}$ (rule of selection of the sign of $\bar{u}$ is the same as for the radial impulse). In the case of a tangential impulse, the geometrical addition of velocities is replaced by the algebraic.

Problem 8.1. A satellite is moving in a circular Earth orbit r_0 with circular velocity v_{ci}. Determine the radial velocity impulse $\bar{u}$ as a result of whose application the satellite passes into an elliptical orbit with assigned perigee distance r_π.

Solution. Let us look at the positive and negative impulses (Figs. 18 and 19). In both cases the applied impulse $\bar{u}$ is perpendicular to $\bar{v}_{ci}$, therefore the resultant velocities, even for their different directions, will be the same: $v = \sqrt{v_{ci}^2 + u^2}$. Therefore, /83

in both cases the direction of velocity is positive ($\sqrt{v_{ci}^2 + u} >$ $> v_{ci}$) and the satellite passes into an elliptical orbit with semi-major axis $\alpha > r_0$. The orientation of the orbital line of apsides depends entirely on the direction of $\bar{u}$.

To determine the velocity at the perigee of the new orbit π, let us write the integral of areas for two points on this orbit -- -- perigee π and the launch point M_0:

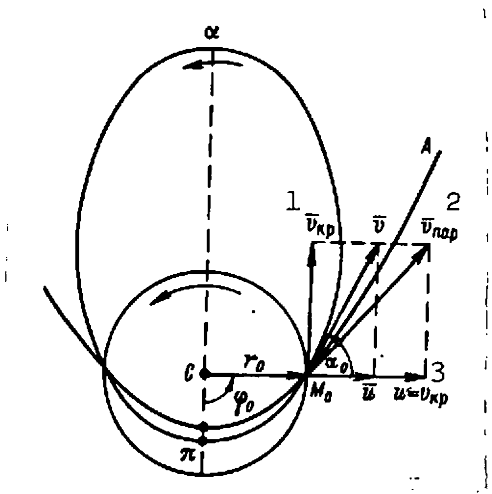

Fig. 18

Key: 1. $\bar{v}_{ci}$
2. $\bar{v}_{par}$
3. v_{ci}

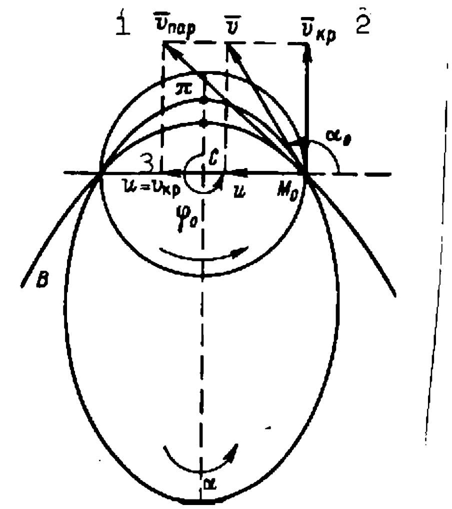

Fig. 19

Key: 1. $\bar{v}_{par}$
2. $\bar{v}_{ci}$
3. v_{ci}

$$\bar{r}_\pi \times \bar{v}_\pi = \bar{r}_0 \times \bar{v}, \quad r_\pi v_\pi = r_0 v \sin\alpha_0 .$$ /83

From vector geometry it follows that $\sin\alpha_0 = v_{ci}/v$, so that we have

$$v_\pi = v_{vi} \frac{r_0}{r_\pi} . \quad (8.1)$$

We can reach the same relation by assuming that the launch point belongs simultaneously to both orbits, as a result of which the constants of the areas of these orbits are

$\bar{c} = \bar{r}_\pi \times \bar{v}_\pi = \bar{r}_0 \times \bar{v} = \bar{r}_0 \times \bar{v}_{ci}$, and $c = r_\pi v_\pi = r_0 v_{ci}$.

Let us write the integral of energy for these two points:

$$v_\pi^2 = v_{ci}^2\left(\frac{r_0}{r_\pi}\right)^2 = \frac{2\mu_\delta}{r_\pi} - \frac{\mu_\delta}{a}, \quad v^2 = v_{ci}^2 + u^2 = \frac{2\mu_\delta}{r_0} - \frac{\mu_\delta}{a}. \quad (8.2)$$

Canceling out $\mu_{\oplus}/\alpha$ by means of the relation

$$\frac{\mu_{\oplus}}{\alpha} = \frac{2\mu_{\oplus}}{r_{\pi}} - v_{ci}^{2}\left(\frac{r_0}{r_{\pi}}\right)^{2}, \tag{8.3}$$

we get for u^2

$$u^2 = \frac{2\mu_{\oplus}}{r_0} - v_{ci}^{2} - \left[\frac{2\mu_{\oplus}}{r_{\pi}} - v_{ci}^{2}\left(\frac{r_0}{r_{\pi}}\right)^{2}\right].$$

Replacing v_{ci}^2 by its value $\mu_{\oplus}/r_0$, let us find

$$u^2 = \frac{\mu_{\oplus}}{r_0} - \frac{2\mu_{\oplus}}{r_{\pi}} + \frac{\mu_{\oplus} r_0}{r_{\pi}^2},$$

so that

$$u^2 = \frac{\mu_{\oplus}}{r_{\pi}}\left(\frac{r_{\pi}}{r_0} - 2 + \frac{r_0}{r_{\pi}}\right) = \frac{\mu_{\oplus}}{r_{\pi}}\left(\sqrt{\frac{r_0}{r_{\pi}}} - \sqrt{\frac{r_{\pi}}{r_0}}\right)^{2},$$

or

$$u^2 = \frac{\mu_{\oplus}}{r_0}\left[1 - 2\frac{r_0}{r_{\pi}} + \left(\frac{r_0}{r_{\pi}}\right)^{2}\right] = \frac{\mu_{\oplus}}{r_0}\left(1 - \frac{r_0}{r_{\pi}}\right)^{2} = v_{ci}^{2}\left(1 - \frac{r_0}{r_{\pi}}\right)^{2}.$$

Finally, for the desired impulse we have /84

$$u = \pm v_{1.ci\,\pi}\left(\sqrt{\frac{r_0}{r_{\pi}}} - \sqrt{\frac{r_{\pi}}{r_0}}\right), \tag{8.4}$$

or

$$u = \pm v_{ci}\left(\frac{r_0}{r_{\pi}} - 1\right), \tag{8.4'}$$

where $v_{1.ci\,\pi} = \sqrt{\frac{\mu_{\oplus}}{r_{\pi}}}$ is the familiar local circular velocity at the perigee of the new orbit.

Since the expression in the parentheses is positive ($r_0 > r_{\pi}$), the sign in front of it corresponds to the sign of impulse u.

Thus, the given distance r_{π} can be attained for both directions of the radial impulse vector. The orientation of the

elliptical orbit here differs. We can easily see that in the limiting case for the impulse $|u| = v_{ci}$ ($v_{ci} > 0$) parabolas A and B are formed ($v = \sqrt{v_{ci}^2 + u^2} = \sqrt{2}\, v_{ci} = v_{par}$) with mutually opposite direction of focal axes.

Interestingly, according to Eq. (8.3) the length of the semi-major axis of the new elliptical orbit does not depend on the direction of the impulse (for its fixed value). Thus, both new elliptical orbits (see Figs. 18 and 19) have identical semi-axes and eccentricities (for equal r_π), although the orientations differ. Correspondingly, the periods of revolution of the satellites in these orbits are also the same (see problem 8.2).

Problem 8.2. The satellite moves in a near-Earth circular orbit with radius r_0, making one revolution in the time T. As a result of a radial velocity impulse u being applied, it passes over into an elliptical orbit. Determine the period of revolution of the satellite T_1 in the elliptical orbit.

Solution. In problem 8.1 it was shown that the length of the semi-major axis of an elliptical orbit α formed due to the application of the radial velocity impulse and, therefore, the period of revolution T_1, do not depend on the direction of impulse $\bar{u}$: for a fixed impulse $\bar{u}$ these quantities prove to be equal both for the positive and negative impulses (see Figs. 18 and 19). The value of α can be determined from the integral of energy using the second formula of (8.2):

$$v^2 = v_{ci}^2 + u^2 = \frac{\mu_\delta}{r_0} + u^2 = \mu_\delta\left(\frac{2}{r_0} - \frac{1}{\alpha}\right),$$ /85

whence

$$\alpha = \frac{\mu_\delta r_0}{\mu_\delta - r_0 u^2} = \frac{r_0}{1 - r_0 u^2/\mu_\delta}, \qquad \alpha\Big|_{u=0} = r_0 . \tag{8.5}$$

Assuming α to be known, let us use Kepler's third law:

$$\frac{T^2}{T_1^2} = \frac{r_0^3}{\alpha^3} = \left(1 - \frac{r_0}{\mu_\delta}\, u^2\right)^3 ,$$

whence

$$T_1 = T\left(1 - \frac{r_0}{\mu_\delta} u^2\right)^{-3/2}. \tag{8.6}$$

For the satellite of a planet or star whose gravitational parameter μ is unknown, by means of the relations of circular motion we can cancel out r_0/μ from Eq. (8.6), then we have

$$v_{ci}^2 = \frac{\mu}{r_0} = \left(\frac{2\pi r_0}{T}\right)^2, \quad \frac{r_0}{\mu} = \left(\frac{T}{2\pi r_0}\right)^2,$$

so that

$$T_1 = T\left[1 - \left(\frac{uT}{2\pi r_0}\right)^2\right]^{-3/2}. \tag{8.7}$$

Problem 8.3. The problem conditions are the same as in problems 8.1 and 8.2. Determine how variation in impulse u with any sign influences the orientation of the focal axis of an ellipse relative to the fixed direction at the launch point M, that is, establish a relation between u and the true anomaly (polar angle) ϕ of the launch point.

Solution. As follows from geometrical relations (Figs. 18 and 19), variation in impulse u influences the direction of v, and therefore, the change in the angle α_0 = arcctg u/v_{ci}. In problem 7.4 we established the formula (7.10) relating angle ϕ_0 with the angle $\beta_0 = /2 - \alpha_0$ (angle with the horizon): /86

$$\operatorname{tg}\varphi_0 = \frac{\operatorname{tg}\beta_0}{1 - r_0/p} = \frac{\operatorname{ctg}\alpha_0}{1 - r_0/p} = \frac{u}{v_{ci}(1 - r_0/p)}.$$

As we can readily see, the parameter of the elliptical orbit is equal to the parameter of the initial circular orbit, since parameter p is defined by the formula $p = c^2/\mu$, and the constant of areas c, as indicated in problem 8.1, is the same for both orbits, namely:

$$c = r_0 v_{ci} = r_0 v \sin\alpha_0 = r_0\sqrt{v_{ci}^2 + u^2}\,\sin\alpha_0,$$

where

$$\alpha_0 = \frac{v_{ci}}{v} = \frac{v_{ci}}{\sqrt{v_{ci}^2 + u^2}}$$

so that

$$p = \frac{c^2}{\mu_\delta} = \frac{r_0^2 v_{ci}}{\mu_\delta} = r_0 \ .$$

This value of parameter p could have been obtained at once by the formula $p = \alpha\,(1 - e^2)$ for the circular orbit when $\alpha = r_0$, and $e = 0$. The value of the parameter $p = r_0 = \alpha\ (1 - e^2)$ is retained also for the entire family of elliptical trajectories corresponding to the set of values $|u| < v_{ci}$. In this case the orbital elements α and e of each ellipse differ.

Thus, assuming p to be known, we get $\mathrm{tg}\ \phi_0 \to \infty$, that is, ϕ_0 in general does not depend on $\bar{u}$, and thus, does not depend on v, and here we have a fixed value $\phi_0 = \pi/2$, or $\phi_0 = 3\pi/2$. The first value of ϕ_0 (see Fig. 18) corresponds to $u > 0$, and the second (see Fig. 19) -- $u < 0$. This means that the perigee of any elliptical orbit corresponding to a radial impulse with specific sign is situated at a fixed angular distance from M_0. When the sign of u is changed, the direction Earth center - perigee is reversed. The directions of the focal axes of the orbits coincide here for u with any sign.

A variation in the elements of the orbital family can be determined by Eqs. (8.5) for α and from $p = \alpha\ (1 - e^2)$ for e:

$$\alpha = \frac{r_0}{1 - \frac{r_0}{\mu_\delta} u^2} = \frac{r_0}{1 - \left(\frac{u}{v_{ci}}\right)^2} ,$$

$$e = \sqrt{1 - \frac{p}{\alpha}} = \sqrt{1 - \frac{r_0}{\alpha}} = \sqrt{\left(\frac{u}{v_{ci}}\right)^2} = \frac{|u|}{v_{ci}} > 0 .$$

Hence it is clear that α and e depend not on the sign of u, since /87 u^2 or $|u|$ appears in the formulas, but only on its value. As u is increased, we observe α and e to increase. As a result, there is a lowering of the perigee, which is evident from the formula

Two parabolas formed when $|u| = v_{ci}$ so that $v = v_{ci}\sqrt{2} = v_{par}$ are the limiting case of elliptical trajectories (except for the case of initial circle). The formulas for α and e in the particular cases ($u = 0$ and $|u| = v_{ci}$) correspond to a circle ($\alpha = r_0$ and $e = 0$) and a parabola ($\alpha \to \infty$, $e \to 1$). The perigee distances of both parabolas are the same: $r_\pi = \frac{1}{2}p = \frac{1}{2}r_0$. Thus, by specifying a set of radial impulses $|u| \leq v_{ci}$, we observe a gradual evolution of the trajectories from the parabola A to parabola B, that is, an evolution of parabola A - ellipse - circle - ellipse - parabola B.

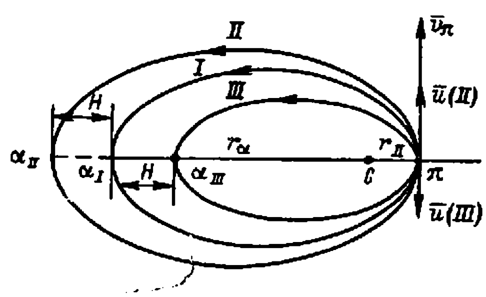

Fig. 20

Problem 8.4. A satellite moves in an elliptical Earth orbit I (Fig. 20) with specified radius-vectors of perigee r_π and apogee r_α. Determine the tangential velocity impulse u plight of the perigee for which the apogee altitude is increased by a given amount H.

Solution. The apogee altitude can be increased only by applying at the perigee a positive velocity impulse $\bar{u} = u\tau^0$ ($u > 0$), during which the orbital velocity $v = v_\pi + u$ ($\bar{v} = \bar{v}_\pi + \bar{u}$). Here the perigee of orbit I is the perigee also of the new orbit II, and the semi-major axis $\alpha_I = 1/2\,(r_\pi + r_\alpha)$ is increased to $\alpha_{II} = 1/2\,(r_\pi + r_\alpha + H)$. This occurs due to an increase in the orbital velocity and follows directly from the integral of energy $v^2 = \mu_{\delta}(2/r_\pi - 1/\alpha_{II})$.

From the integrals of energy written for the perigees of both orbits we have

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$$v_\pi = \sqrt{\frac{2\mu_\delta}{r_\pi} - \frac{\mu_\delta}{\alpha_I}} = \sqrt{\frac{2\mu_\delta}{r_\pi} - \frac{2\mu_\delta}{r_\pi + r_\alpha}} = \sqrt{\frac{2\mu_\delta r_\alpha}{r_\pi(r_\pi + r_\alpha)}},$$

$$v = v_\pi + u = \sqrt{\frac{2\mu_\delta}{r_\pi} - \frac{\mu_\delta}{\alpha_{II}}} = \sqrt{\frac{2\mu_\delta}{r_\pi} - \frac{2\mu_\delta}{r_\pi + r_\alpha + H}} = \sqrt{\frac{2\mu_\delta(r_\alpha + H)}{r_\pi(r_\pi + r_\alpha + H)}}, \qquad (8.8)$$

whence

$$u = \sqrt{\frac{2\mu_\delta}{r_\pi}}\left(\sqrt{\frac{r_\alpha + H}{r_\pi + r_\alpha + H}} - \sqrt{\frac{r_\alpha}{r_\pi + r_\alpha}}\right). \qquad (8.9)$$

When the apogee altitude is reduced by the altitude H (see orbit III in Fig. 20), we must bear in mind that the negative impulse $\bar{u}$ ($u < 0$) is directed opposite to $\bar{v}_{\pi}$, so that there is a reduction in the perigee velocity from v_{π} to $v = v_{\pi} + u < v_{\pi}$. The perigees of orbits I and III coincide, and the value of the semi-axis $\alpha_{I} = 1/2\ (r_{\pi} + r_{\alpha})$ drops to the value $\alpha_{III} = 1/2\ (r_{\pi} + r_{\alpha} - H)$. We get the formula of the corresponding impulse by replacing H in Eq. (8.9) by (-H):

$$u = \sqrt{\frac{2\mu_{\delta}}{r_{\pi}}}\left(\sqrt{\frac{r_{\alpha} - H}{r_{\pi} + r_{\alpha} - H}} - \sqrt{\frac{r_{\alpha}}{r_{\pi} + r_{\alpha}}}\right). \tag{8.9'}$$

Problem 8.5. A spacecraft is moving in an Earth orbit ($r = 150 \cdot 10^{6}$ km) with circular heliocentric velocity $v_{ci} = 29.78$ km/sec. What tangential velocity impulse must be imparted to this craft for its new orbit at its aphelion to be tangent to the circular orbit of Mars ($r_1 = 228 \cdot 10^{6}$ km). Determine the same for the case of light to Venus ($r_2 = 108 \cdot 10^{6}$ km) (Fig. 13).

Solution I. Let us use the formulas of the preceding problem, by writing it in general form for $\mu = \mu_{\odot}$ and $r = r_{\pi} = r_{\alpha}$:

$$u = \sqrt{\frac{2\mu_{\odot}}{r}}\left(\sqrt{\frac{r \pm H}{2r \pm H}} - \frac{1}{\sqrt{2}}\right) = \sqrt{2}\, v_{ci}\left(\sqrt{\frac{r \pm H}{2r \pm H}} - \frac{1}{\sqrt{2}}\right),$$

where $\sqrt{2}\, v_{ci} = v_{par} = 42.11$ km/sec. For the case of flight to the orbit of Mars,

$$r_1 > r, \quad u(\mathrm{I}) > 0, \quad H = r_1 - r, \quad u = 42.11\left(\sqrt{\frac{r + H}{2r + H}} - \right.$$

$$\left. - \frac{1}{\sqrt{2}}\right) = 42.11\left(\sqrt{\frac{r_1}{r + r_1}} - \frac{1}{\sqrt{2}}\right) = 42.11\left(\sqrt{\frac{228}{378}} - \frac{1}{\sqrt{2}}\right) = 2.95 \text{ km/sec},$$ /89

and for the case of flight to the orbit of Venus

$$r_2 < r, \quad u(\mathrm{II}) < 0, \quad H = r - r_2, \quad u = 42.11\left(\sqrt{\frac{r - H}{2r - H}} - \frac{1}{\sqrt{2}}\right) =$$

$$= 42.11\left(\sqrt{\frac{r_2}{r + r_2}} - \frac{1}{\sqrt{2}}\right) = 42.11\left(\sqrt{\frac{108}{258}} - \frac{1}{\sqrt{2}}\right) = -2.53 \text{ km/sec}.$$

After tangency to the orbit of Mars or Venus, the spacecraft will continue to move in an elliptical trajectory, returning to the point of impulse application. The transfer to the orbit of Mars or Venus is possible only when an additional impulse is applied at the point of tangency.

Solution II. Let us write a series of formulas characterizing the transfer of the spacecraft from one elliptical orbit into another. The integral of energy is of the form

$$v^2 = \mu_\odot\left(\frac{2}{r} - \frac{1}{\alpha}\right) = v_{ci}^2\left(2 - \frac{r}{\alpha}\right). \tag{8.10}$$

If it is assumed that the craft, traveling at velocity v_0, acquires this velocity by an additional tangential impulse u ($v = v_0 + u$) being imparted to it, then

$$u = v - v_0 = v_{ci}\sqrt{2 - \frac{r}{\alpha}} - v_0, \tag{8.11}$$

where α is the semi-major axis of the new orbit. In particular, if the initial orbit were circular, $v_0 = v_{ci}$, and the corresponding impulse would be

$$u = v_{ci}\left(\sqrt{2 - \frac{r}{\alpha}} - 1\right), \quad v_{ci} = \sqrt{\frac{\mu_\odot}{r}}. \tag{8.11'}$$

If velocity v is assumed to be equal to the velocity of the craft before application of the impulse as the result of which it acquires the final velocity $v_{fin} = v + u$, then we have

$$u = v_{fin} - v = v_{fin} - v_{ci}\sqrt{2 - \frac{r}{\alpha}}, \tag{8.12}$$

where α is the semi-major axis of the initial orbit. For the final circular orbit with $v_{fin} = v_{ci}$, we have /90

$$u = v_{ci}\left(1 - \sqrt{2 - \frac{r}{\alpha}}\right), \quad v_{ci} = \sqrt{\frac{\mu_\odot}{r}}. \tag{8.12'}$$

For Mars, $r/\alpha = 3/3.78 = 0.794$, $u = 29.78\ (\sqrt{2 - 0.794} - I = 2.95$ km/sec. For Venus, $r/\alpha = 3/2.58 = 0.860$, $u = 29.78 \times (I - \sqrt{2 - 0.860}) = -2.53$ km/sec.

Problem 8.6. A satellite is moving in a circular orbit with radius r, making one revolution in the time T. When a tangential velocity impulse u is applied, it passes into an elliptical orbit. Determine the period of revolution T_1.

Solution. Let us derive the formula for the period of revolution in the elliptical orbit for a tangential impulse in and direction. Based on Kepler's third law $T^2/T_1^2 = r^3/\alpha^3$ and and the integral of energy, we have

$$\frac{v^2}{v_{ci}^2} = 2 - \frac{r}{\alpha} \qquad \text{and} \qquad \frac{T^2}{T_1^2} = \frac{r^3}{\alpha^3} = \left(2 - \frac{v^2}{v_{ci}^2}\right)^3.$$

With the tangential impulse $v^2 = (v_{ci} + u)^2$ (u with any sign) where $v_{ci} = 2\pi r/T$, so that

$$T_1 = T\left[1 - \frac{uT}{\pi r} - \left(\frac{uT}{2\pi r}\right)^2\right]^{-3/2}.$$

Calculate the flight time to the orbits of Mars and Venus (Fig. 13). In problem 8.5 it was determined that a tangential impulse u = 2.95 km/sec > 0, was needed for a flight to the orbit of Mars, while an impulse u = - 2.53 km/sec < 0 was needed for a flight to the orbit of Venus. Adopting for the Earth's orbit T = 365.3 days = $3.15 \cdot 10^7$ sec, and r = $150 \cdot 10^6$ km, we get that for a flight to the orbit of Mars

$$T_1 = 365{,}3\left[1 - \left(\frac{2{,}95 \cdot 3{,}15 \cdot 10^7}{3{,}14 \cdot 1{,}5 \cdot 10^8}\right) - \frac{1}{4}\left(\frac{2{,}95 \cdot 3{,}15 \cdot 10^7}{3{,}14 \cdot 1{,}5 \cdot 10^8}\right)^2\right]^{-3/2} =$$

$$= 365{,}3\left[1 - 0{,}197 - 0{,}001\right]^{-3/2} = 365{,}3\sqrt{\left(\frac{1}{0{,}792}\right)^3} =$$

$$= 365{,}3 \cdot 1{,}411 = 515{,}1 \text{ days}, \quad \tau_1 = \frac{1}{2} T_1 = 257{,}6 \text{days}.$$ /91

Under the same conditions, in the case of a flight to the orbit of Venus we have

$$T_1 = 365{,}3\left[1 - \left(\frac{2{,}53 \cdot 3{,}15 \cdot 10^7}{3{,}14 \cdot 1{,}5 \cdot 10^8}\right) - \frac{1}{4}\left(\frac{2{,}53 \cdot 3{,}15 \cdot 10^7}{3{,}14 \cdot 1{,}5 \cdot 10^8}\right)^2\right]^{-3/2} =$$

$$= 365.3\left[1 + 0.168 - 0.001\right]^{-3/2} = 365.3\sqrt{\left(\frac{1}{1.165}\right)^3} =$$

$$= 365.3 \cdot 0.798 = 291.9 \text{ days}, \tau_1 = \frac{1}{2} T_1 = 146.0 \text{ days}.$$

The same values of τ were obtained in problem 6.10 by another method.

Problem 8.7. A spacecraft moving in a circular satellite orbit around the Earth must be launched from it, receiving a tangential velocity impulse, and must be inserted into hyperbolic orbit with given velocity v_∞. At which radius r of the initial circular orbit will the impulse be smallest?

Solution. Insertion into hyperbolic orbit can be effected only by applying a positive tangential impulse u (Fig. 21), when the circular velocity of the craft rises to the hyperbolic velocity $v = v_{ci} + u > v_{par} = \sqrt{2}\, v_{ci}$. From the integral of energy for hyperbolic motion $v^2 - v_{par}^2 = v^2 - 2v_{ci}^2 = v_\infty^2 = h$ it follows that $(v_{ci} + u)^2 - 2v_{ci}^2 = v_\infty^2$, so that $u = \sqrt{2v_{ci}^2 + v_\infty^2} - v_{ci}$. Since $dv_{ci}/dr = d(\sqrt{\mu_♁/r} \neq 0)/dr$, the condition $du/dr = 0$ corresponding to the desired minimum of impulse u

$$\frac{du}{dr} = \left(\frac{2v_{ci}}{\sqrt{2v_{ci}^2 + v_\infty^2}} - 1\right)\frac{dv_{ci}}{dr} = 0,$$

can be satisfied only when $v^2 = 2v_{ci}^2 = v_{par}^2 = \sqrt{2\mu_♁/r}$.

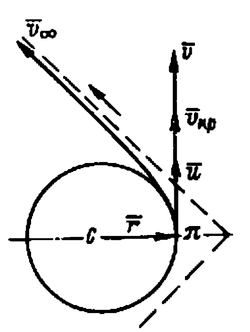

Fig. 21

This corresponds to the impulse /92
$u = v_{ci}$, hyperbolic velocity $v = 2v_{ci}$, and radius-vector $r = 2\mu_♁/v_\infty^2 = 2\mu_♁/h$.

Problem 8.8. A transfer, which is called a transfer in a Hohmann ellipse, is effected from an initial circular orbit J with radius r_1 (Fig. 22 a) to the final circular coplanar orbit F with radius r_2. At point A of orbit

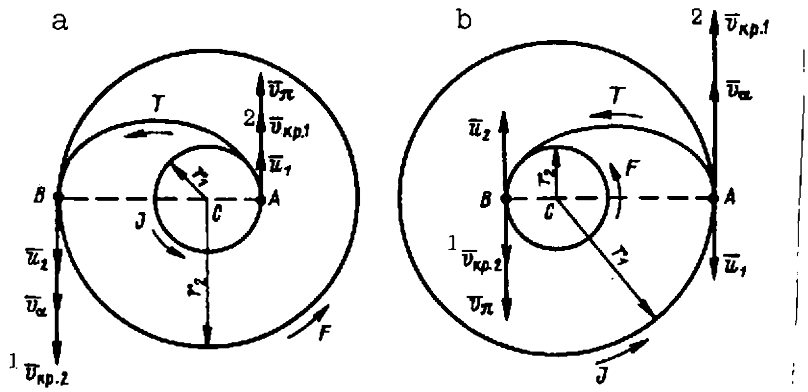

Fig. 22

Key: 1. $\bar{v}_{ci.2}$

2. $\bar{v}_{ci.1}$

J the spacecraft receives a positive tangential impulse $\bar{u}_1$. As a result, motion continues along the ellipse T ($\alpha > r_1$) to point B (orbital apocenter T), where a second positive impulse $\bar{u}_2$ inserts the craft into its final orbit F ($r_2 > \alpha$). Determine the total incremental velocity $u = u_1 + u_2$ in transferring from one orbit to the other and determine at which ratios of r_1 to r_2 will the ratio u to the initial velocity $v_{ci.1}$ be at a maximum (the problem of transfer via a Hohmann ellipse).

Solution I. Let us look at the initial craft velocity at point A as it moves in the circular orbit $v_{ci.1} = \sqrt{\mu/r_1}$. The value of the velocity after the increment $v_\pi = v_{ci.1} + u_1$ at this same point, which is the pericenter of the transfer orbit T with semi-axis $\alpha = 1/2\,(r_1 + r_2)$, can be determined from the integral of areas:

$$v_\pi = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{\alpha}\right)} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right)} = \sqrt{\frac{2\mu}{r_1} \cdot \frac{r_2}{r_1 + r_2}},$$

so that /93

$$v_\pi = v_{ci.1} + u_1 = \sqrt{\frac{2\mu}{r_1} \cdot \frac{r_2}{r_1 + r_2}}.$$

Thus, for transfer to orbit T, the craft must be given the tangential impulse

$$u_1 = v_\pi - v_{ci.1} = \sqrt{\frac{2\mu}{r_1} \cdot \frac{r_2}{r_1+r_2}} - \sqrt{\frac{\mu}{r_1}} > 0 .$$

When impulse $\bar{u}_2$ is applied at the apocenter of the transfer orbit, that is, at point B, the velocity after the increment $v_{ci.2} = v_\alpha + u_2$, where

$$v_{ci.2} = \sqrt{\frac{\mu}{r_2}} ,$$

$$v_\alpha = \sqrt{\mu\left(\frac{2}{r_2} - \frac{2}{r_1+r_2}\right)} = \sqrt{\frac{2\mu}{r_2} \cdot \frac{r_1}{r_1+r_2}} = v_\pi \frac{r_1}{r_2} ,$$

so that

$$u_2 = v_{ci.2} - v_\alpha = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} \cdot \frac{r_1}{r_1+r_2}} > 0 .$$

Let us set up the formula for the overall velocity increment $u = u_1 + u_2$, by representing it as

$$\frac{u}{v_{ci.1}} = \sqrt{\frac{r_1}{r_2}} - \sqrt{\frac{2 r_1^2}{r_2(r_1+r_2)}} + \sqrt{\frac{2 r_2}{r_1+r_2}} - 1 .$$

Introducing the notation $R = r_2/r_1 > 1$, we can write

$$\frac{U}{v_{ci.1}} = \sqrt{\frac{1}{R}} - \sqrt{\frac{2}{R(1+R)}} + \sqrt{\frac{2R}{1+R}} - 1 = \sqrt{\frac{2R}{1+R}}\left(1 - \frac{1}{R}\right) + \sqrt{\frac{1}{R}} - 1 > 0 . \quad (8.13)$$

Let us determine the desired maximum of this quantity by the formula

$$\frac{d}{dR}\left(\frac{u}{v_{ci.1}}\right) = \frac{1}{R^2}\left(\frac{2R}{1+R}\right)^{1/2} + \frac{R-1}{R(1+R)^2}\left(\frac{2R}{1+R}\right)^{-1/2} - \frac{1}{2R^{3/2}} = 0 ,$$

whence there follows the cubic equation $R^3 = 15R^2 - 9R - 1 = 0$, /94
which has a real root $R_{max} = r_2/r_1 = 15.58176$. When $R > R_{max}$, $u/v_{ci.1}$ decreases. A plot of this quantity as a function of R is shown in Fig. 23. From this figure it follows that the energy outlays necessary, for example, to reach the orbit of Pluto from the Earth's orbit, are greater than to reach the orbit of Mars, but smaller than to reach Jupiter's orbit. The energy outlays for heliocentric Earth-Saturn and Earth-Uranus flights are about the same as for flights to Jupiter and Neptune.

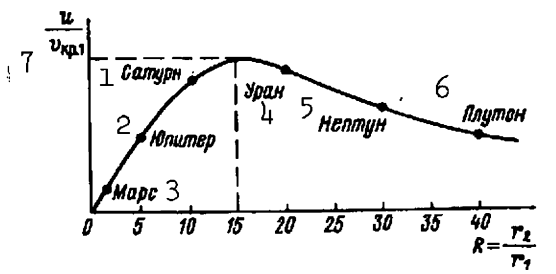

Fig. 23

Key: 1. Saturn
2. Jupiter
3. Mars
4. Uranus
5. Neptune
6. Pluto
7. $v_{ci.1}$

Solution II. To determine u_1 and u_2, we can use Eqs. (8.11') and (8.12'):

$$R = \frac{r_2}{r_1}, \quad u_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2R}{1+R}} - 1\right), \quad u_2 = -\sqrt{\frac{\mu}{r_2}}\left(\sqrt{\frac{2}{1+R}} - 1\right),$$

after which again get

$$\frac{u}{v_{ci.1}} = \frac{(u_1 + u_2)\sqrt{r_1}}{\sqrt{\mu}} = \sqrt{\frac{2R}{1+R}}\left(1 - \frac{1}{R}\right) + \sqrt{\frac{1}{R}} - 1 .$$

This formula is valid for any R, and when $R < 1$, $u < 0$, and when $R > 1$ (as in this case), $u > 0$.

Problem 8.9. Using the formulas in problem 8.8, calculate the total velocity impulse $u = u_1 + u_2$ needed for a transfer via a Hohmann ellipse from the circular Earth orbit ($v_{ci.1} = 29.78$ km/sec, $r_1 = 150 \cdot 10^6$ km) to the circular orbit of Mars ($r_2 = 228 \cdot 10^6$ km). Characterize the velocity change as the craft transfers from orbit to the other. /95

Solution. Let us use Eq. (8.13) for the calculations, where

$$R = \frac{r_2}{r_1} = \frac{228}{150} = 1.52, \quad \text{so that} \quad \frac{u}{v_{ci.1}} = \sqrt{\frac{2R}{1+R}}\left(1 - \frac{1}{R}\right) + \sqrt{\frac{1}{R}} - 1 = \sqrt{\frac{3.04}{2.52}}(1 - 0.66) + \sqrt{0.66} - 1 = 0.376 + 0.811 - 1 = 0.187,$$

and $u = 0.187 \cdot v_{ci1} = 0.187 \cdot 29.78 = 5.57$ km/sec.

The distribution of velocity impulses as the craft transfers from the Earth's orbit to the orbit of Mars is carried out as follows. On being launched from the Earth's orbit ($v_{ci.1}$ = 29.78 km/sec), the craft given a positive impulse u_1 = 2.95 km/sec (see problem 8.5) sufficient for the transfer ellipse T at the aphelion to touch the orbit of Mars. The elliptical transfer is characterized by these velocities (see Fig. 22):

$$v_A = v_\pi = v_{ci.1} + u_1 = 29.78 + 2.95 = 32.73 \text{ km/sec},$$
$$v_B = v_\alpha = v_\pi \frac{r_\pi}{r_\alpha} = 32.73 \frac{150}{228} = 21.53 \text{ km/sec}.$$

The second positive impulse $u_2 = u - u_1$ = 5.57-2.95 = 2.62 km/sec is important to the craft at the aphelion of the transfer ellipse B and inserts the craft into the circular Mars orbit with circular velocity $v_{ci.2} = v_B + u$ = 21.53 + 2.62 = 24.15 km/sec. This circular heliocentric velocity in the orbit of Mars can be obtained by the formula

$$v_{ci.2} = \sqrt{\frac{\mu_\odot}{r_\sigma}} = \sqrt{\frac{1327 \cdot 10^8}{228 \cdot 10^6}} = \sqrt{583.2} = 24.15 \text{ km/sec}.$$

Problem 8.10. Determine the total velocity increment in transfer via a Hohmann ellipse from initial circular orbit with radius r_1 to a final circular orbit with smaller radius r_2 (Fig. 22 b).

Solution. For point A, which after application of a negative velocity impulse $\bar{u}_1$ becomes the apocenter of the Hohmann transfer ellipse T, we have

$$v_{ci.1} = \sqrt{\frac{\mu}{r_1}},$$

/96

$$v_\alpha = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)} = \sqrt{\frac{2\mu}{r_1} \cdot \frac{r_1}{r_1 + r_2}},$$

$$u_1 = v_\alpha - v_{ci.1} = \sqrt{\frac{2\mu}{r_1} \cdot \frac{r_2}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}} < 0.$$

When the impulse is again applied at pericenter B, we have

$$v_{ci.2} = \sqrt{\frac{\mu}{r_2}}, \quad v_\pi = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a}\right)} = \sqrt{\frac{2\mu}{r_2} \cdot \frac{r_1}{r_1 + r_2}},$$

$$u_2 = v_{ci.2} - v_* = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} \cdot \frac{r_1}{r_1 + r_2}} < 0.$$

Thus, we again obtain Eq. (8.13):

$$\frac{u}{v_{ci.1}} = \sqrt{\frac{2R}{1+R}}\left(1 - \frac{1}{R}\right) + \sqrt{\frac{1}{R}} - 1 < 0,$$

where $R = \frac{r_2}{r_1} < 1.$

Problem 8.11. Using the formulas in problem 8.10, determine the total velocity impulse $u = u_1 + u_2$ needed to transfer a spacecraft from the circular Earth orbit ($v_{ci.1}$ = 29.78 km/sec, r_1 = = $150 \cdot 10^6$ km) to the circular Venus orbit ($r_2 = 108 \cdot 10^6$ km).

Solution. Setting

$$R = \frac{r_2}{r_1} = \frac{108}{150} = 0.720,$$

in Eq. (8.13), we get

$$u/v_{ci.1} = \sqrt{\frac{1.440}{1.720}}\,(1 - 1.389) + \sqrt{1.389} - 1 = -0.176,$$

and the total impulse $u = 0.176\, v_{ci.1} = -0.176 \cdot 29.78 = -5.24$ km/sec.

The distribution of impulses occurs as follows. During a launch from Earth orbit ($v_{ci.1}$ = 29.78 km/sec), the craft is given a negative impulse $u_1 \cong -2.53$ km/sec (see Problem 8.5) sufficient for the transfer ellipse T to touch the orbit of Venus.

Further, we have $v_A = v_\alpha = v_{ci.1} + u_1$ = 27.25 km/sec, $v_B = v_\pi$ = $v_\alpha\, r_\alpha / r_\pi = 27.25 \cdot 150/108$ = 37.77 km/sec. /97

The second negative impulse $u_2 = u - u_1 = -5.24 + 2.53 =$ = 2.71 km/sec is imparted to the craft at the perihelion B and inserts it into the Venus orbit: $v_{ci.2} = v_B + u_2 = 37.77 - 2.71 =$ = 35.06 km/sec. It is precisely this circular velocity along the

Venus orbit that we find from the formula

$$v_{ci.2} = \sqrt{\frac{\mu_\odot}{r_2}} = \sqrt{\frac{1327\cdot 10^8}{108\cdot 10^6}} = \sqrt{1229} = 35.06 \text{ km/sec.}$$

Problem 8.12. Using the results of problem 818, determine the total velocity increment for the limiting case of transfer to an "infinitely distant" final circular orbit ($r_2 >> r_1$) (Fig. 24 a).

Solution. Determine the limit of Eq. (8.18) as $R=\frac{r_2}{r_1}\to\infty$:

$$\lim_{R\to\infty}\frac{u}{v_{ci.1}} = \lim_{R\to\infty}\left[\sqrt{\frac{2R}{1+R}}\left(1-\frac{1}{R}\right)+\sqrt{\frac{1}{R}}-1\right] = \sqrt{2}-1.$$

Thus, the total velocity increment tends to the limit

$$u = u_1 + u_2 = (\sqrt{2}-1)\,v_{ci.1} \approx 0.41\, v_{ci.1} > 0,$$

that is, from the standpoint of energy outlays this transfer is equivalent to the departure of the body "to infinity" in a parabolic trajectory whose pericenter is at the point of application of the unit velocity impulse $u = (\sqrt{2}-1)\, v_{ci.1} = v_{par} - v_{ci.1}$. Solving the inverse problem -- on return "from infinity" to the initial orbit J, which is equivalent to the problem of the departure of the point "to infinity" with final orbit F, we obtain the formula for the total negative impulse $u = u_1 + u_2 = -(\sqrt{2}-1)\, v_{ci.2} = -0.41\, v_{ci.2} < 0$.

Problem 8.13. Determine the overall velocity increment needed to transfer a craft along a three-impulse bielliptical trajectory (Fig. 24 b) when its motion originates initially along ellipse T_1 to point B', where it is given a second positive impulse $\bar{u}_2$ (see condition of problem 8.8), and then along ellipse T_2 to point C, where this craft receives a third, negative impulse u_3 inserting it into the final orbit F.

Solution. To determine the impulses u_1, u_2, and u_3, let us use successively Eqs. (8.11'), (8.11), and (8.12'): /98

$$u_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2Q}{1+Q}}-1\right),\quad R=\frac{r_2}{r_1},\quad Q=\frac{r_{B'}}{r_1},$$

$$u_2 = \sqrt{\frac{\mu}{r_{B'}}}\sqrt{2-\frac{2r_{B'}}{r_2+r_{B'}}} - v_{B'} = \sqrt{\frac{\mu}{r_1}}\left[\sqrt{\frac{2R}{Q(Q+R)}}-\sqrt{\frac{2}{Q(1+Q)}}\right],$$

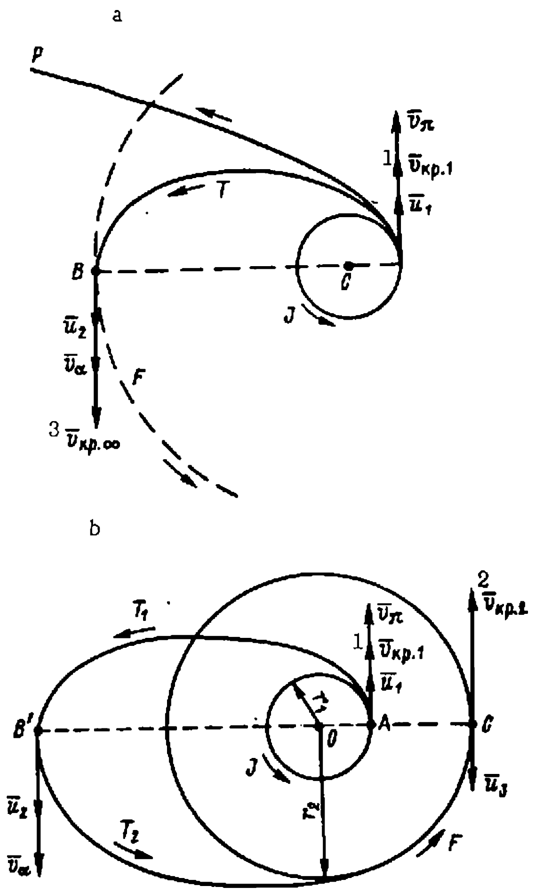

Fig. 24

Key: 1. $\bar{v}_{ci.1}$
2. $\bar{v}_{ci.2}$
3. $\bar{v}_{ci.\infty}$

$$u_3 = \sqrt{\frac{\mu}{r_{B'}}}\left(1 - \sqrt{2 - \frac{2r_2}{r_2 + r_{B'}}}\right) = \sqrt{\frac{\mu}{r_1}}\left[-\sqrt{\frac{2Q}{R(Q+R)}} + \sqrt{\frac{1}{R}}\right],$$ /99

since according to the theorem of areas we have

$$v_{B'} = \left(\sqrt{\frac{\mu}{r_1}} + u_1\right)\frac{r_1}{r_{B'}} = \sqrt{\frac{\mu}{r_1}} \cdot \frac{1}{Q}\sqrt{\frac{2Q}{1+Q}}.$$

Finally,

$$\frac{u}{v_{ci.1}} = \frac{(u_1 + u_2 + u_3)\sqrt{r_1}}{\sqrt{\mu}} = \sqrt{\frac{2Q}{1+Q}} + 1 + \sqrt{\frac{2R}{Q(Q+R)}} - \sqrt{\frac{2}{Q(1+Q)}} + \sqrt{\frac{1}{R}} - \sqrt{\frac{2Q}{R(Q+R)}}. \quad (8.14)$$

We can easily see that from Eq. (8.14) when $R \equiv Q$ $(r_2 \equiv r_{B'})$ Eq. (8.13) follows as a particular case.

Problem 8.14. Determine the energy needed to rotate the plane of the circular orbit of a craft moving along it at velocity $v_1 = v_{ci} = \sqrt{\frac{\mu}{r}}$ by 90°.

Solution. Suppose a spacecraft is moving along a circular orbit with velocity $\bar{v}_1$ and at some moment of time receives the increment $\Delta\bar{v}_1$ perpendicular to the radius-vector $\bar{r}$ at point M. Then the resultant velocity $\bar{v}_2 = \bar{v}_1 + \Delta\bar{v}_1$ will also be perpendicular to $\bar{r}$, and if $|\bar{v}_2| > |\bar{v}_1|$, the point M of application of the impulse will become the pericenter of the new elliptical orbit (Fig. 25). Here the plane of the new orbit will be inclined to the plane of the initial orbit at an angle i (the angle between the vector is $\bar{v}_1$ and $\bar{v}_2$).

From the vector triangle we have by the theorem of cosines

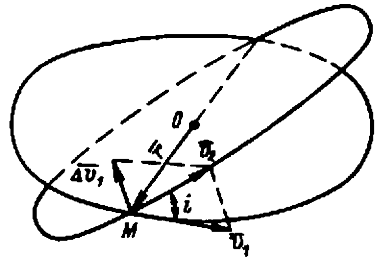

Fig. 25

$$\Delta v_1 = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \cos i}\,.$$

To conserve the circular velocity v_1, the craft must be given the impulse $\Delta v_1 = 2 v_1 \sin\frac{i}{2}$.

In this case the inclination of the new orbit to the plane of the initial orbit i is arbitrary. In the limiting case (when $i = 90°$ and $v_1 = v_2$) we obtain /100

$$\Delta v_1 = v_1\sqrt{2} = \sqrt{\frac{2\mu}{r}}\,. \tag{8.15}$$

Let us perform the calculation. Let the spacecraft move in a circular orbit of the Earth with circular heliocentric velocity v_1 - 29.78 kn/sec (artificial planet). For the craft to be inserted into a circular heliocentric orbit whose radius is equal to the radius of the Earth's orbit and which is inclined to the plane of the Earth's orbit by $i = 10°$ there is required the impulse

$$\Delta v_1 = 2 v_1 \sin\frac{i}{2} = 2\cdot 29.78\cdot 0.087 = 5.18 \text{ km/sec},$$

and for insertion into a polar heliocentric orbit ($i = 90°$), the impulse

$$\Delta v_1 = v_1\cdot\sqrt{2} = \sqrt{2}\cdot 29.78 = 42.11 \text{ km/sec}$$

is required. Thus, the energy necessary to rotate the plane of the circular orbit 90°, $m(\Delta v)^2/2$, is equal to the energy needed to provide the circular velocity $\sqrt{\frac{2\mu}{r}}$ at the point of application of the tangential velocity impulse, that is, to transfer the craft from a circular orbit to a parabolic orbit lying in the same plane.

CHAPTER NINE /100

SPHERE OF ACTION. PROBLEMS OF THIRD ESCAPE VELOCITY

The sphere of action is an arbitrary geometrical concept employed to "delimit" two gravity fields. The term sphere of action of a lesser gravitating mass relative to a greater refers to an imaginary surface within which it is useful to assume the lesser mass m as the central body, and the greater mass M as the perturbing body. The relation of the accelerations of a point with mass m relative to a point with mass M (m < M) at the boundary of the sphere of action is as follows:

$$\frac{\text{Perturbing acceleration due to M}}{\text{Acceleration due to central mass m}} = \frac{\text{Perturbing acceleration due to m}}{\text{Acceleration due to central mass M}}$$

If we neglect the perturbing acceleration, we can conditionally assume the sphere of action to be a sphere of equal zero acceleration delimiting the two gravity fields. The radius of the sphere of action must be determined from the formula $\rho = \Delta(m/M)^{2/5}$,

where Δ is the distance between the masses. From this formula let us find the radius of the sphere of action of the Earth relative to the Sun, assuming $m = M_{\oplus} = 6 \cdot 10^{27}$ g, $M = M_{\odot} = 1.97 \cdot 10^{33}$ g, and $\Delta = 149.6 \cdot 10^6$ km:

$$\rho = 149.6 \cdot 10^6 \left(\frac{6 \cdot 10^{27}}{1.97 \cdot 10^{33}}\right)^{2/3} = 929{,}900 \approx 930{,}000 \text{ km.}$$ /101

Similarly, we can determine, for example, the radius of the Moon's sphere of action relative to the Earth (ρ = 66,000 km) or the radius of the sphere of action of the Sun relative to the center of our Galaxy ($\rho = 9 \cdot 10^{12}$ km).

In the problems presented below, we will allow for the dimensions of the sphere of action of the Earth only for geocentric motion of a point. For heliocentric motion we can neglect the dimensions of the sphere of action and assume that a point at the boundary of the Earth's sphere of action has the same heliocentric circular velocity as the Earth:

$$v_{1.ci/\odot} = v_{\delta/\odot} = \sqrt{\frac{\mu_\odot}{149.6\cdot 10^6}} = \sqrt{\frac{1327\cdot 10^8}{149.6\cdot 10^6}} = 29.78 \text{ km/sec}.$$

If the point (rocket), on departing beyond the boundary of the Earth's sphere of action, has sufficient residual geocentric velocity $v_{bo.roc/\underset{o}{+}}$, its resultant heliocentric velocity is determined from the theory of velocity additions:

$$v_{roc/\odot} = v_{1.ci/\odot} + v_{bo.roc/\underset{o}{+}}. \tag{9.1}$$

The positive direction of velocities always is associated with the true direction of the Earth's orbital motion, which is taken as positive.

Problem 9.1. Determine the velocity at which a rocket must be launched from the Earth's surface $|v_{1a}|$ for the rocket to stop at the boundary of the sphere of action ($v_{bo.roc/\underset{o}{+}} = 0$). How much time will this flight take? Characterize the further heliocentric motion of the rocket.

Solution. Let us write the interval of area of the launch point of the Earth's surface and at the boudary of the sphere of action:

$$\tilde{h} = v_{cr}^2 - \frac{2\mu_\delta}{R_\delta} = v^2_{bo.roc/\underset{o}{+}} - \frac{2\mu_\delta}{\rho}, \tag{9.2}$$

whence when $v_{bo.roc/\underset{o}{+}} = 0$ and $\rho = 930{,}000$ km, we determine the constant of energy $\tilde{h}$: $\tilde{h} = -\frac{2\mu_\delta}{\rho} = -\frac{2\cdot 398{,}600}{930{,}000} = -0.857$ km^2/sec^2 < 0,

such that the trajectory of the geocentric arrival at the boundary is an ellipse (stopping at a finite distance is not possible for parabolic motion). The velocity of the geocentric launch is /102

$$|v_{1a}| = \sqrt{\tilde{h} + \frac{2\mu_\delta}{R_\delta}} = \sqrt{\tilde{h} + v_{II}^2} = \sqrt{-0.857 + (11.19)^2} = 11.15 \text{ km/sec} \tag{9.3}$$

and it is close to the parabolic velocity v_{II} = 11.19 km/sec.

The flight time to the boundary should be determined from Kepler's equation. Let us find, first, from the integral of area α: $v_{1a}^2 = v_\pi^2 = \mu_{\oplus}(2/R_{\oplus} - 1/\alpha)$, whence

$$\alpha = \frac{R_{\oplus}\mu_{\oplus}}{2\mu_{\oplus} - R_{\oplus}v_{1a}^2} = \frac{6370 \cdot 398600}{2 \cdot 398600 - 6370\,(11{,}15)^2} = 470{,}200 \text{ km.}$$

The eccentricity of the ellipse $e = 1 - \frac{r_\pi}{\alpha} = 1 - \frac{6370}{470200} = 0{,}986$. From the formula

$$\rho = \alpha(1 - e\cos E) = 930{,}000 \text{ km}$$

we get cos E = - 0.992, sin E = 0.126, E = 170°44' = 3.015, and e sin E = 0.124, so that Kepler's equation becomes

$$\tau = \frac{\alpha^{3/2}}{\sqrt{\mu_{\oplus}}}(E - e\sin E) = \frac{(4{,}702 \cdot 10^5)^{3/2}}{\sqrt{398600}} \times$$

$$\times\ (3.015 - 0.124) = 1.476 \cdot 10^6 \text{ sec} =$$

$$= 410 \text{ hours} = 17.08 \text{ days.}$$

Let us characterize the further, heliocentric motion of a rocket. Within the frame of reference of the above assumptions it can be stated that a rocket, losing geocentric velocity, "always" remains at the boundary of the Earth's sphere of action and will move together with the sphere, that is, together with the Earth, around the Sun with circular velocity $v_{ci/\odot}$ = 29.78 km/sec. Actually, this motion will originate along a trajectory close to the Earth's orbit, and with a velocity along this trajectory that will be determined by the distance $149 \cdot 10^6$ km ± 930,000 km to the Sun.

Problem 9.2. A rocket is launched from the Earth's surface with the second escape velocity v_{II} = 11.19 km/sec and will move along a parabolic trajectory intersecting the Earth's sphere of action. Determine with what residual parabolic velocity the rocket arrives at the boundary of the Earth's sphere of action and how much time will the flight take to reach the boundary? Characterize the heliocentric motion of the rocket after departing from the Earth's sphere of action.

Solution. Let us determine the residual parabolic geocentric velocity at the boundary of the sphere of action for given initial /103

launch conditions:

$$|v_{bo.roc/\oplus}| = |v_{1.par}| = \sqrt{\frac{2\mu_\oplus}{\rho}} = \sqrt{\frac{2\cdot 398600}{929900}} = 0.927 \text{ km/sec}.$$

The flight time to the boundary of the sphere of action can be found from the formula for the flight time along the hyperbola from problem 6.15:

$$\tau(\rho) = (3\sqrt{\mu_\oplus})^{-1} \sqrt{2\rho - p}\,(\rho + p),$$

from which when ρ = 929,900 km, $p = 2r_\pi = 2R_\oplus$ = 12,740 km we get

$$\tau = (3\sqrt{398600})^{-1} \sqrt{2\cdot 929900 - 12740}\,(929900 - 12740) =$$

$$= 0.243\cdot 10^6 \text{ sec} = 67.60 \text{ hours} = 2.81 \text{ days}.$$

The character of the further, heliocentric motion of the rocket is determined by the direction $\bar{v}_{ci.roc/\oplus}$, which in this problem is not fixed. In any case, the theorem of [velocity] addition (9.1) enables us to obtain either the increase in the heliocentric velocity within the limits $29.78 < v_{roc/\odot} \leq 29.78 +$ $+ 0.93$ $(29.78 < v_{roc/\odot} \leq 30.71)$ km/sec or else its decrease within the limits $29.78 - 0.93 \leq v_{roc/\odot} < 29.78$ $(28.85 \leq v_{roc/\odot} < 29.78$ km/sec. In the first case the rocket will arrive at the elliptical trajectory whose semi-major axis is greater than the axis of the Earth's orbit, where the point of departure from the Earth's sphere of action becomes the perihelion of the elliptical orbit, and the aphelion is on the other side of the Sun. When $v_{roc/\odot} = 30.71$ km/sec, the orbital plane of the rocket coincides with the Earth's orbital plane. In the second case, the semi-major axis is smaller than the radius of the Earth's orbit, the departure point becomes the aphelion, and the coplanar orbit is formed when $v_{roc/\odot} = 28.85$ km/sec. The direction of motion in both cases coincides with the direction of Earth motion (direct orbits). We note that this consideration is known to be valid for any residual velocities in the interval $0 < |v_{bo.roc/\oplus}| \leq 0.93$.

The problem of the addition of heliocentric velocities at the boundary of the Earth sphere of action and the determination of

the trajectory of the rockets heliocentric motion is identical to the problem of the motion of a rocket moving along an initial circular orbit under the influence of the additional velocity impulse $\bar{u}$ or arbitrary direction and magnitude (see Chapter Eight). In this case the role of impulse $\bar{u}$ is played by residual velocity /104 $\bar{v}_{bo.roc/\oplus}$.

Problem. 9.3. With what minimum geocentric velocity must a rocket be launched from the Earth's surface for its heliocentric trajectory to be tangent at its aphelion to the orbit of Mars? Solve this same problem for the case of tangency to the orbit of Venus.

Solution. In the case of the heliocentric trajectory being tangent to the orbit of Mars, the minimum launch velocity $|v_{1a}|$ is the velocity at which the direction coinciding to the residual velocity $\bar{v}_{bo.roc/\oplus}$ coincides with the direction of the Earth's circular velocity so that $\bar{v}_{bo.roc/\oplus} > 0$, and on the basis of the theorem of [velocity] addition (9.1) we get the algebraic sum (the orbits are coplanar). Reducing this problem to the problem of a positive tangential velocity impulse $\bar{u} \equiv \bar{v}_{bo.roc/\oplus} > 0$, let us use the value $u = v_{bo.roc/\oplus} = 2.95$ km/sec (it is shown in problem 8.5 that this velocity impulse is necessary and sufficient for the elliptical orbit of the rocket at its aphelion to be tangent to the orbit of Mars). Writing the integral of energy (9.2), we can determine the constant of energy for the geocentric trajectory:

$$\tilde{h} = v^2_{bo.roc/\oplus} - \frac{2\mu_\delta}{\rho} = (2.95)^2 - \frac{2 \cdot 398600}{929\,900} = 8.702 - 0.857 = 7.845 > 0.$$

Hence it follows that the geocentric trajectory of departure is a hyperbola. Indicentally, this also follows from a comparison of the value of the residual velocity at the boundary of the sphere of action with the corresponding value of the local parabolic velocity (see problem 9.2):

$$v_{bo.roc/\oplus} = 2.95 \text{ km/sec}, \quad v_{1.par} = 0.93 \text{ km/sec}.$$

Let us determine the required hyperbolic launch velocity:

$$|v_{1a}| = \sqrt{\tilde{h} + v_{\text{I}}^2} = \sqrt{7.845 + (11.19)^2} = 11.53 \quad \text{km/sec}.$$

It corresponds to the resultant heliocentric perihelion velocity at the boundary $v_\pi = v_{roc/\odot} - 29.78 + 2.95 = 32.73$ km/sec, so that the heliocentric trajectory is an ellipse.

In the case when the heliocentric trajectory is tangent to the orbit of Venus, the residual velocity corresponding to the minimum launch velocity $|v_{1a}|$ must be directed so that there is /105
an algebraic subtraction of velocities and so that the resultant aphelion velocity $v_\alpha = v_{roc/\odot} = 29.78 - 2.53 = 27.25$ km/sec, where $u = v_{bo.roc/\oplus} = -2.53$ km/sec is the negative tangential impulse necessary and sufficient for the elliptical trajectory to be tangent at its aphelion to the orbit of Venus (see problem 8.5). The constant of energy corresponds to the geocentric trajectory

$\tilde{h} = v^2_{bo.roc/\oplus} - \frac{2\mu_\delta}{\rho} = 6.401 - 0.857 = 5.544 > 0$, so that the geocentric trajectory of departure as before is a hyperbola. Actually, by comparing velocities it follows that $|v_{bo.roc/\oplus}| =$ 2.53 km/sec, and $|v_{1.par}| = 0.93$ km/sec. The minimum hyperbolic launch velocity here

$$|v_{1a}| = \sqrt{\tilde{h} + v_{\text{II}}^2} = \sqrt{5.544 + (11.19)^2} = 11.43 \text{ km/sec},$$

so that the heliocentric trajectory is an ellipse.

Note that in both these cases the elliptical orbits are coplanar orbits of Earth, Mars, and Venus and have the same direction of motion, coinciding with the direction of orbital motion of the Earth, that is, they are direct orbits.

Problem 9.4. What must the minimum geocentric launch velocity of a rocket from the Earth's surface be $|v_{1a}|$ for the rocket to acquire the heliocentric velocity $v_{roc/\odot}$ needed to leave the solar system along a parabolic trajectory. (Problem of third escape velocity.)

Solution. The parabolic heliocentric velocity in the Earth's orbit is $v_{1.par} = 42.11$ km/sec (see problem 5.6). Therefore, for the rocket to leave the solar system along a parabola it is necessary and sufficient for the direction of the geocentric escape velocity $\bar{v}_{bo.roc/\oplus}$ to ensure that the quality $\bar{v}_{roc/\odot} =$ $= \bar{v}_{1.par}$ is satisfied. Here the minimum launch velocity corre-

sponds to the case of the algebraic sum of velocities based on Eq. (9.1), which ensures the direct orientation of motion along a parabolic trajectory coplanar to the orbits of the Earth and the other planets (Fig. 26).

Let us find the residual boundary velocity:

$$v_{bo.roc/\oplus} = v_{roc/\odot} = 42.11 - 29.78 = 12.33 \text{ km/sec.}$$

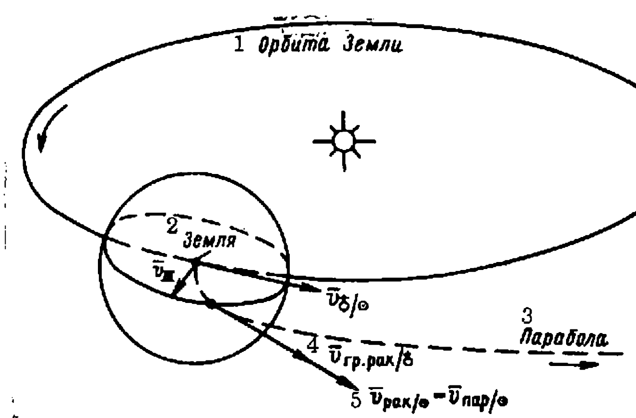

Fig. 26

Key: 1. Earth orbit
2. Earth
3. Parabola
4. $\bar{v}_{bo.roc}$
5. $\bar{v}_{roc/\odot} = \bar{v}_{par/\odot}$

Let us calculate the constant of energy $\tilde{h}$ from the integral of energy for the hyperbolic escape trajectory: /106

$$\tilde{h} = v^2_{bo.roc/\oplus} - \frac{2\mu_\delta}{\rho} = (12{,}33)^2 - \frac{2 \cdot 398600}{930000} =$$

$$= 151.18 \text{ km}^2/\text{sec}^2 > 0.$$

To this value $\tilde{h}$ there corresponds the hyperbolic launch velocity

$$|v_{1a}| = \sqrt{\tilde{h} + v_{II}^2} = \sqrt{151{,}18 + (11{,}19)^2} =$$

$$= 16.62 \text{ km/sec.}$$

The minimum velocity attained is called the third escape velocity v_{III}. The launch of a rocket from the Earth's surface with this velocity ensures realization of the heliocentric escape velocity.

This problem on the escape of a rocket from the solar system along a direct parabolic orbit is identical to the problem of the application of a positive tangential impulse $u \equiv v_{bo.roc/\oplus} = 12.33$ km/sec. This velocity is the upper limit to the velocities $|v_{bo.roc/\oplus}|$ with which the resultant heliocentric orbit will be elliptical. By comparing this result with the result of problem 9.2, we can conclude that with variation in velocity, having any direction, within the limit $0 < |v_{bo.roc/\oplus}| < 12.33$ km/sec, the heliocentric orbit will be an ellipse.

Based on the formulas in problems 9.1-9.4 we can solve the problem of the "falling" of a point for the Sun or the problem /107
of an insertion along a parabola to the inverse orbit. In the problem of "falling" toward the Sun it is assumed that the rocket at the boundary of the Earth's sphere of action has a zero heliocentric velocity, that is, that it is "at rest" relative to the Sun. Under the effect of its gravity force the rocket begins to move toward the Sun rectilinearly (along a heliocentric radius-vector). Here $v_{bo.roc/\oplus} = - v_{\oplus/\odot} = - 29.78$ km/sec, $|v_{1a}| \approx \approx \sqrt{v^2_{bo.roc/\oplus} + v^2_{II}} = 31.81$ km/sec, which corresponds to the rectilinear "falling" trajectory lying in the plane of the Earth's orbit. The theoretical time of motion can be determined from Kepler's third law, by examining rectilinear motion as the limiting case of motion along a severely elongated ellipse with semi-major axis $a \approx 150 \cdot 10^6/2 = 75.10^6$ km, the period of motion along which is 130 days, so that the time of flight to the sun is 65 days.

According to the problem of insertion into an inverse circular orbit $v_{bo.roc/\oplus} = - 2\, v_{1.ci/\odot} = - 59.56$ km/sec, $|v_{1a}| \approx 60.60$ km/sec, where motion occurs along the Earth's orbit in the direction opposite to the true direction of its motion, such that in half a year the rocket will again encounter the Earth. According to the problem of escape from the solar system along a hyperbola for motion in the direction opposite the motion of the Earth (inverse parabolic orbit), we have $v_{bo.roc/\oplus} = - v_{1.par} - v_{1.ci} = - 42.11 - 29.78 = - 71.89$ km/sec, $|v_{1a}| = 72.80$ km/sec.

Problem 9.5. Determine the launch velocity $|v_{1a}|$ required for a rocket to be inserted into a circular heliocentric orbit whose radius is equal to the radius of the Earth's orbit, and whose plane is perpendicular to the plane of the Earth's orbit.

Solution. The problem has two solutions: flight to the "north pole" ($i = 90°$) and flight to the "south pole" ($i = 270°$). In both cases motion will occur with the same circular heliocentric velocity $v_{ci/\odot} = 29.78$ km/sec as the motion of the Earth, so that for an equilateral vector triangle (Fig. 27), the formula established in problem 8.13 for the velocity impulse obtains:

$$|v_{bo.roc/\oplus}| = 2 v_{\oplus/\odot} \sin \frac{i}{2} = \sqrt{2}\, |v_{1.ci/\odot}| = |v_{1.par/\odot}| =$$

$$= 42.11 \text{ km/sec.}$$

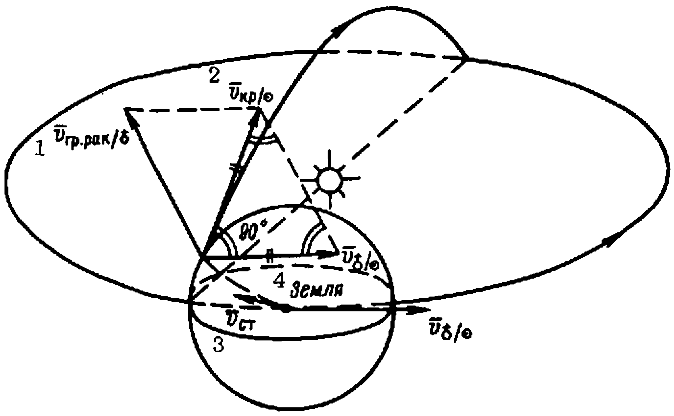

Fig. 27

Key: 1. $\bar{v}_{bo.roc/♁}$
2. $\bar{v}_{vi/\odot}$
3. $\bar{v}_{la}$
4. Earth

Fig. 27 shows the case of flight to the "north pole," when the vector $\bar{v}_{bo.roc/♁}$ makes an angle of 135° with the vector of the Earth's velocity $\bar{v}_{♁/\odot}$ (for flight to the "south pole" this angle is 225°). In both cases the launch velocity at the Earth's surface is /108

$$|v_{la}| \approx \sqrt{v^2_{bo.roc/♁} + v^2_{II}} =$$

$$= 43.46 \text{ km/sec.}$$

CHAPTER TEN

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TWO-BODY PROBLEM. THIRD KEPLER'S LAW GENERALIZED

In the previous sections we considered the limited two-body problem. In solving this problem it was assumed that the mass of a space object is sufficiently small so that its attraction of the central body does not affect the motion of the latter. However, in the case when natural celestial bodies interact, the central body under the influence of the other body executes some motion, which in turn is reflected in the motion of the former body. As a result, both bodies execute Keplerian motions relative to their common mass center (barycenter) with equal periods of revolution. In the problems given below we have examined some questions of the motion of the "attracting" mass m in the case when the acceleration of the central attracting mass M under the influence of attraction by mass m cannot be neglected.

Problem 10.1. Two free points with masses m and M ($m < M$) are moving under the influence of gravitational forces [mutual attraction forces]. Determine the law of motion of mass m relative to mass M.

Solution. Let us consider the motion of points m and M in an absolute inertial coordinate system (Fig. 28). For the radius-vectors we have /109

$$\bar{\rho}_m = \bar{\rho}_M + \bar{r}, \quad \frac{d^2\bar{\rho}_m}{dt^2} = \frac{d^2\bar{\rho}_M}{dt^2} + \frac{d^2\bar{r}}{dt^2}.$$

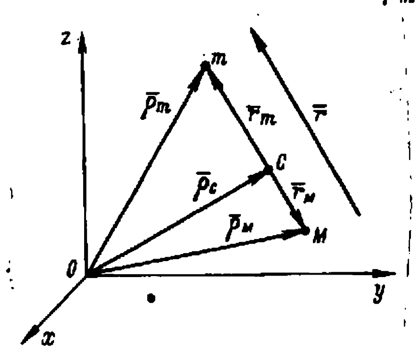

Fig. 28

As a result of the action of the attractive force, mass M will tend to shift relative to the origin of coordinates in the direction $\bar{r}$, and mass m will tend to move in the direction opposite to it. Here the equations of absolute motion of both masses are:

$$M \frac{d^2\bar{\rho}_M}{dt^2} = \frac{fmM}{r^3}\bar{r}, \quad m \frac{d^2\bar{\rho}_m}{dt^2} = -\frac{fmM}{r^3}\bar{r}.$$

Dividing both parts of the equalities by the mass m, we get

$$\frac{d^2\bar{\rho}_M}{dt^2} = fm\frac{\bar{r}}{r^3}, \quad \frac{d^2\bar{\rho}_m}{dt^2} = -fM\frac{\bar{r}}{r^3}. \qquad (10.1)$$

The equation of motion of mass m relative to M becomes

$$\frac{d^2\bar{r}}{dt^2} = \frac{d^2\rho_m}{dt^2} - \frac{d^2\rho_M}{dt^2} = -f(M+m)\frac{\bar{r}}{r^3}, \qquad (10.1')$$

or

$$\frac{d^2\bar{r}}{dt^2} + f(M+m)\frac{\bar{r}}{r^3} = 0. \qquad (10.1'')$$

By comparing the resulting equation (10.1") with the equation of absolute motion (1.5) for a "nonattracting" point from the restricted two-body problem, we conclude that in this case relative motion will occur according to the same laws as absolute motion, but the gravitational parameter $\mu = f\ (M + m)$. In other words, the gravitating satellite m will move about the central body M /110 as if a "nonattracting" satellite m were moving around a central body with mass M + m.

From the foregoing it follows that the formulas according to which in Chaps. 1-9 we determined the various kinematic and dynamic characteristics of motion (velocity, time of motion, energy, and so on) can be used for determining these same characteristics of motion also in the general two-body problem. But in this case the gravitational parameter must not be $\mu = fM$, but must have the new value $\mu = f\ (M + m)$.

Problem 10.2. Two homogeneous spheres with radii R_1 and R_2 will begin to move from a state of rest under the influence of forces of mutual attraction. Determine with which relative velocity v_r these spheres will collide if the initial distance between their centers is L, and the mass are m_1 and m_2.

Solution. Let us examine the case of rectilinear relative motion in the general two-body problem. As indicated in problem 10.1, in this case relative motion can be replaced by the absolute motion of one (any) sphere in the field of attraction of the other (fixed) sphere with mass $m_1 + m_2$. Assuming that the spheres attract as material points, we will consider that, for example, the point with mass m_2 drops toward the point with mass $m_1 + m_2$, where this dropping occurs from altitude L to altitude $R_1 + R_2$.

The force of attraction of mass $m_1 + m_2$ acting on point m_2 is $- fm_2 (m_1 + m_2)r^{-2}$ (we associate the positive direction with the radius-vector of point m_2, drawn from $m_1 + m_2$), and the positive work done by this attractive force in moving the mass m_2 over this section is $fm_2 (m_1 + m_2) (1/R_1 + R_2 - 1/L)$. The theorem on change in the kinetic energy of mass m_2 when it is in absolute motion and has the initial velocity $v_0 = 0$ is of the form

$$\frac{m_2 v_r^2}{2} = f m_2 (m_1 + m_2)\left(\frac{1}{R_1 + R_2} - \frac{1}{L}\right),$$

so that

$$v_r = \sqrt{2f\,(m_1 + m_2)\left(\frac{1}{R_1 + R_2} - \frac{1}{L}\right)}. \qquad (10.2)$$

If in this formula we assume $m_2 \ll m_1 = M_{\oplus}$, $R_1 = R_{\oplus}$, and $R_2 = 0$, then again we get the formula for the absolute velocity /111
of dropping of the point onto the Earth's surface from altitude H:

$$v = \sqrt{2fM_{\oplus}\left(\frac{1}{R_{\oplus}} - \frac{1}{R_{\oplus} + H}\right)} = \sqrt{\frac{2fM_{\oplus}H}{R_{\oplus}(R_{\oplus} + H)}} = \sqrt{\frac{2gR_{\oplus}H}{R_{\oplus} + H}},$$

that we derived in problem 1.9. Based on problems 1.8 and 1.9, it can be concluded that "receding" spheres must have this same velocity v_r (as the initial velocity) in order to be separated by the given distance L.

Problem 10.3. Two free points whose masses are m and M move under the influence of forces of mutual attraction. Determine the motion of the points relative to their common mass center (barycenter) C (Fig. 28).

Solution. The position of the barycenter C, lying along the line connecting the masses m and M, is defined by vector $\bar{\rho}_C$. Thus, $(M + m)\ \bar{\rho}_C = M\bar{\rho}_M + m\bar{\rho}_m$.

If we let $\bar{r}_M$ and $\bar{r}_m$ stand for the vectors determining the positions of M and m relative to C, then $\bar{\rho}_M = \bar{\rho}_c + \bar{r}_M$; $\bar{\rho}_m = \bar{\rho}_c + \bar{r}_m$, so that we can write $(M + m)\bar{\rho}_c = M(\bar{\rho}_c + \bar{r}_M) + m(\bar{\rho}_c + \bar{r}_m)$ and

$$M\bar{r}_M + m\bar{r}_m = 0. \qquad (10.3)$$

Since in this case the radius-vector of mass m measured relative to the radius-vector of mass M is of the form $\bar{r} = \overline{Mm} = \bar{r}_m - \bar{r}_M$, we can write $\bar{r}_M = \bar{r}_m - \bar{r}$, and $\bar{r}_m = \bar{r} + \bar{r}_M$. The successive substitution of these expressions into (10.3) gives $M(\bar{r}_m - \bar{r}) + m\bar{r}_m = 0$, $M\bar{r}_M + m(\bar{r} + \bar{r}_M) = 0$, whence $(M + m)\bar{r}_m = M\bar{r}$, $(M + m)\bar{r}_M = -m\bar{r}$ and

$$\bar{r} = \frac{M+m}{M}\bar{r}_m = -\frac{M+m}{m}\bar{r}_M. \qquad (10.3')$$

Thus, the orbits described by masses M and m around the common mass center C are similar to each other and similar to the orbit described by a one mass around the other. /112

Let us look at the elliptical orbits of two bodies with masses M and m shown in Fig. 29 a. For specificity, we assume M = 2m, which can correspond, for example, to the case of a binary star. Both bodies describe about their barycenter C, as about a focus, similar ellipses (with equal eccentricities), while continually remaining along a line drawn through the barycenter, on the opposite sides of it. The mass m describes an ellipse that is twice as large as the ellipse described by mass M. The relation $u_m/M = a_M/m$ deriving from (10.3) corresponds to this motion.

Let us evaluate this same phenomenon from the standpoint of an observer situated on a large star M for which it is fixed. Taking from Fig. 29 a the distances mM for each of the instants 1, 2, ..., 6 and plotting them in the corresponding direction, let us schematically represent the orbit of the star m relative to M (Fig. 29 b). Obviously, the major axis of the new orbit /113

must be equal to the sum of the axes of the orbits of both stars in their barycentric motion (Fig. 29 a), so that the similarity of the three orbits is evident. In the general form, the relation

$$\frac{a_m}{M} = \frac{a_M}{m} = \frac{a_M + a_m}{M + m}$$

corresponds to this construction.

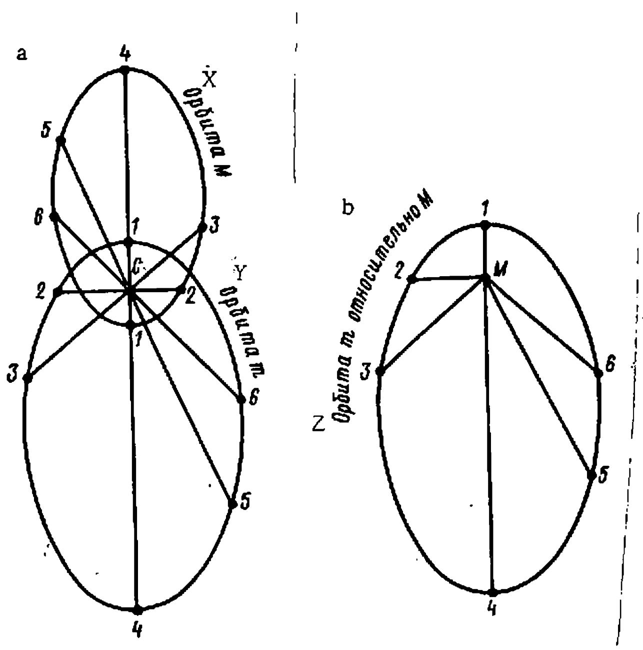

Fig. 29

Key: X. Orbit of M
Y. Orbit of m
Z. Orbit of m relative to M

Let us write out the equations of motion relative to the mass center with reference to the directions $\bar{r}_M$ and $\bar{r}_m$:

$$m\frac{d^2\bar{r}_m}{dt^2} = -fmM\frac{\bar{r}}{r^3}, \quad M\frac{d^2\bar{r}_m}{dt^2} = -fmM\frac{\bar{r}}{r^3},$$

so that

$$\frac{d^2\bar{r}_m}{dt^2} = -fM\frac{\bar{r}}{r^3}, \quad \frac{d^2\bar{r}_M}{dt^2} = -fm\frac{\bar{r}}{r^3}.$$

Substituting in place of $\bar{r}$ the expression (10.3') we obtained earlier, we get

$$\frac{d^2\bar{r}_m}{dt^2} = -\frac{fM^3}{(M+m)^2}\cdot\frac{\bar{r}_m}{r_m^3}, \quad \frac{d^2\bar{r}_M}{dt^2} = -\frac{fm^3}{(M+m)^2}\cdot\frac{\bar{r}_M}{r_M^3}. \tag{10.4}$$

Each of these equations has the same form as Eq. (10.1'), so that study of the relative motion of the points of both cases reduces to solving an equation of the form

$$\frac{d^2\bar{r}}{dt^2} + \mu\frac{\bar{r}}{r^3} = 0. \tag{10.5}$$

Thus, motion relative to the barycenter along the ellipses depicted in Fig. 29 a originates according to the same laws as absolute motion, but in this case the gravitational parameters are

$$\mu_m = \frac{fM^3}{(M+m)^2}, \quad \mu_M = \frac{fm^3}{(M+m)^2}. \tag{10.6}$$

Here the motion of mass m relative to mass M obeys the same law, but the parameter $\mu = f\,(M + m)$ (see Eq. (10.1")).

Problem 10.4. What kind of function will obtain between the periods T_i of the revolutions of planets around the Sun and the semi-major axes α_i of their elliptical orbits if the motion of the Sun caused by the attraction of the corresponding planet is taken into account? Give the mathematical formulation of Kepler's third law, generalized. Consider the cases of motion of bodies relative to each other and relative to the barycenter. /114

Solution. It is based on problem 10.1 in which it was established that the relative motion of a mass when one allows for mutual attraction obeys the same laws as absolute motion, but where the gravitational parameter $\mu = f\,(M + m)$. Starting from the fact that for absolute motion the relation between the periods of revolution t_i and the semi-major axes of the planetary orbits α_i is of the form (see problem 6.5):

$$\frac{\alpha_i^3}{T_i^2} = \frac{\mu_\odot}{4\pi^2} = \frac{fM_\odot}{4\pi^2}, \quad T_i = 2\pi\sqrt{\frac{\alpha_i^3}{fM_\odot}}\,. \tag{10.7}$$

it is of the form

$$\frac{\alpha_i^3}{T_i^2} = \frac{f(M_\odot + m_i)}{4\pi^2}, \quad T_i = 2\pi\sqrt{\frac{\alpha_i^3}{f(M_\odot + m_i)}}\,. \tag{10.8}$$

for the relative motion of the i-th planet. For two planets we get, respectively,

$$\frac{\alpha_1^3/T_1^2}{\alpha_2^3/T_2^2} = \frac{f(M_\odot + m_1)}{f(M_\odot + m_2)} = \frac{M_\odot + m_1}{M_\odot + m_2}\,, \tag{10.9}$$

or

$$\frac{\alpha_1^3}{\alpha_2^3} = \frac{T_1^2}{T_2^2} \cdot \frac{1 + m_1/M_\odot}{1 + m_2/M_\odot}\,. \tag{10.10}$$

The Eqs. (10.8)-(10.10) are given the name of Kepler's third law, revised or generalized, in distinction to Eq. (1.11) of absolute motion.

Under barycentric motion in accordance with the equations of motion (10.4)-(10.6), for motions of small mass m_i and large mass

$M_\odot$ we can write, respectively,

$$T_{m_i} = 2\pi\sqrt{\frac{a_{m_i}^3}{\mu_{m_i}}} = 2\pi\sqrt{\frac{a_{m_i}^3\,(M_\odot + m_i)^2}{fM_\odot^3}}, \tag{10.11}$$

$$T_{m_\odot} = 2\pi\sqrt{\frac{a_{m_\odot}^3}{\mu_\odot}} = 2\pi\sqrt{\frac{a_{m_i}^3\,(M_\odot + m_i)^2}{f\,m_i^3}}.$$ /115

Based on the relations for the similar ellipses $a_{m_i}/M_\odot = a_{M_\odot}/m_i$ (see problem 10.3, Fig. 29 a), it can be concluded that the periods of barycentric motions T_{m_i} and $T_{M_\odot}$ are equal to each other.

Reducing the barycentric motion to the motion of a mass m_i relative to the Sun $M_\odot$ (Fig. 29 b), we can again replace the parameters μ_o and μ_{m_i} with the parameter $\mu = f\ (M_o + m_i)$, and the semi-axes α_{M_o} and α_{m_i} by the sum of the semi-axes $\alpha_i = \alpha_{M_o} + \alpha_{m_i}$, as the result of which we again get the formula

$$T_i = 2\pi\sqrt{\frac{(a_{M_\odot} + a_{m_i})^3}{f(M_\odot + m_i)}} = 2\pi\sqrt{\frac{a_i^3}{f(M_\odot + m_i)}}. \tag{10.12}$$

From the relation for similar ellipses

$$\frac{a_{m_i}}{M_\odot} = \frac{a_{M_\odot}}{m_i} = \frac{a_{M_\odot} + a_{m_i}}{M_\odot + m_i}$$

there follows

$$\frac{(a_{M_\odot} + a_{m_i})^3}{M_\odot + m_i} = \left(\frac{a_{m_i}}{M_\odot}\right)^3 (M_\odot + m_i)^2 = \left(\frac{a_{M_\odot}}{m_i}\right)^3 (M_\odot + m_i)^2.$$

The latter equation allows us to state that the period of relative motion coincides with the periods of barycentric motions:

$$T_i = T_{m_i} = T_{M_\odot}.$$

Problem 10.5. Determine the circular velocities of the Moon ($M_☾$) in its circular orbit (r = 384,400 km) and of a rocket (m = 0) executing circular motion in the same orbit.

Solution. To determine the velocity of the rocket, let us use the formulas of the limited two-body problem, so that

$$v_{roc} = \sqrt{\frac{\mu_\delta}{r}} = \sqrt{\frac{398600}{384400}} = 1.018 \text{ km/sec}.$$

To find the Moon's velocity, we must adopt the formula from the general two-body problem and use the formula of rocket velocity, replacing the parameter $\mu_\delta = fM_\delta$ in it by the parameter /116

$\mu_{\delta\zeta} = f(M_\delta + M_\zeta) = \mu_\delta + \mu_\zeta$. Knowing that the gravitational parameter of the Moon $\mu_\zeta = fM_\zeta = 4900 \text{ km}^3/\text{sec}^2$, we have

$$v_\zeta = \sqrt{\frac{\mu_\delta + \mu_\zeta}{r}} = \sqrt{\frac{398600 + 4900}{384400}} = 1.025 \quad \text{km/sec};$$

which means that the Moon will move faster than the rocket. Actually, using Kepler's third law we can determine that the period of revolution of the rocket is greater than the period of revolution of the Moon:

$$T_{roc} = 2\pi\sqrt{\frac{r^3}{\mu_\delta}} = 2\pi\sqrt{\frac{r^3}{fM_\delta}}, \quad T_\zeta = 2\pi\sqrt{\frac{r^3}{\mu_{\delta\zeta}}} = 2\pi\sqrt{\frac{r^3}{f(M_\delta + M_\zeta)}}.$$

By comparing the results we get

$$\frac{T_{roc}}{T_\zeta} = \sqrt{\frac{M_\delta + M_\zeta}{M_\delta}} = \sqrt{1 + \frac{M_\zeta}{M_\delta}} = \sqrt{1 + \frac{1}{81.6}} > 1, \quad (10.13)$$

therefore, $T_{roc} > T_\zeta$.

Problem 10.6. The rocket will move in the circular Moon orbit, where at the initial moment it is at a point diameterally opposite the position of the Moon. How will the mutual disposition of the Moon and rocket change?

Solution. As was shown in problem 10.5, the Moon will move along its orbit faster than the rocket and will overtake it. After a certain time τ has ellapsed, the Moon will catch up to the rocket, that is, the rocket will fall behind it. Let us determine the time τ by means of the formulas from problem 10.5, assuming that the circular motions of the Moon and the rocket are uniform. We can write the angular velocities as follows:

$$\omega_\zeta - \omega_{roc} = \frac{1}{r}(v_\zeta - v_{roc}) = \frac{1.025 - 1.018}{384400} = \frac{0.007}{384400} = 1.82 \cdot 10^{-8} \text{ sec}^{-1}.$$

Obviously, the Moon will catch up to the rocket when its angular distance from the rocket, initially $\phi_0 = \pi$, will reach $\phi = 2\pi$, so that

$$\tau = \frac{\varphi - \varphi_0}{\omega_{☾} - \omega_{roc}} = \frac{\pi}{\omega_{☾} - \omega_{roc}} = \frac{3{,}14 \cdot 10^8}{1{,}82} = 1{,}725 \cdot 10^8 \text{ sec} \approx 6 \text{ years.}$$

We can solve this problem by using Kepler's third law, generalized, by expressing the angular velocity in terms of the periods of revolution: /117

$$\omega_{☾} - \omega_{roc} = 2\pi\left(\frac{1}{T_{☾}} - \frac{1}{T}\right) = \frac{2\pi}{T_{☾}}\left(1 - \frac{T_{☾}}{T_{roc}}\right),$$

$$\tau = \frac{\pi}{\omega_{☾} - \omega_{roc}} = \frac{T_{☾}}{2(1 - T_{☾}/T_{roc})}.$$

Let us transform the expression within the parentheses by using Eq. (10.8):

$$1 - \frac{T_{☾}}{T_{roc}} = 1 - \left(1 + \frac{M_{☾}}{M_{♁}}\right)^{-1/2} \approx 1 - \left(1 - \frac{1}{2} \cdot \frac{M_{☾}}{M_{♁}}\right) = \frac{1}{2} \cdot \frac{M_{☾}}{M_{♁}}.$$

Calculating the mean period of revolution of the Moon

$$T_{☾} = 2\pi\sqrt{\frac{r^3}{\mu_{♁} + \mu_{☾}}} = 2\pi\sqrt{\frac{(384\,400)^3}{398600 + 4900}} = 27{,}32 \text{ days},$$

we get

$$\tau \approx T_{☾}\frac{M_{☾}}{M_{♁}} = 81{,}5\, T_{☾} = 81{,}5 \cdot 27{,}32 = 2{,}23 \cdot 10^3 \text{ days} \approx 6 \text{ years.}$$

CHAPTER ELEVEN /117

MISCELLANEOUS PROBLEMS

Problem 11.1. Determine the useful work that must be done by a rocket engine to lift a spacecraft with mass m to altitude H above the surface of a planet and at this altitude to impart circular and parabolic velocities to it. The weight of the spacecraft at the planetary surface G; planetary radius R, and atmospheric drag can be neglected.

Solution. The work done by the engine A is made up of the work expended in lifting the craft vertically to altitude H and the work needed for the initial-velocity vector to occupy a specific position in the plane of the local horizon. Thus, the total increment in the kinetic energy of the craft $mv^2/2$ (we assume the initial velocity to be zero) is equal to the sum of the work done by the planetary gravity and the useful work done by the engine:

$$\frac{mv^2}{2} = \mu m \left(\frac{1}{R+H} - \frac{1}{R}\right) + A = -\frac{\mu m H}{R(R+H)} + A,$$

where μ is the gravitational parameter of the planet. Obviously, /118
for the craft to be inserted into an orbit with circular velocity

$v_{ci} = \sqrt{\frac{\mu}{R+H}}$, the following work is necessary

$$A_{ci} = \frac{\mu m}{2(R+H)} + \frac{\mu m H}{R(R+H)} = \frac{\mu m (R+2H)}{2R(R+H)},$$

and for insertion with parabolic velocity $v_{par} = \sqrt{\frac{2\mu}{R+H}}$, the following work must be done:

$$A_{par} = \frac{\mu m}{R+H} + \frac{\mu m H}{R(R+H)} = \frac{\mu m}{R},$$

which does not depend on ascent to altitude H.

If the weight of the craft on the planet $G = mg$, where g is the acceleration due to gravity at its surface, using the formula $\mu = fM_{pl} = gR^2$ (see problem 1.6), we get

$$A_{ci} = \frac{GR(R+2H)}{2(R+H)}, \quad \text{and} \quad A_{par} = GR.$$

We can easily calculate that for a craft weighing 5 tons lifted directly from the Earth's surface with parabolic velocity, the desired work A_{par} is $31.85 \cdot 10^9$ kgm.

Problem 11.2. A spacecraft with mass m approaches a planet along a line extending through its center (along the radius-vector). At which altitude from the surface must the engine be fired so that the constant braking force mT it produces will ensure a soft landing (landing with zero velocity)? The velocity of the craft at the instant of engine firing is v_0, the gravitational parameter of the planet is μ, and its radius is R. The attraction of other celestial bodies, atmospheric drag, and change in engine mass can be neglected.

Solution. This motion occurs under the influence of the planetary gravitational force and the braking force of the engine, whose resultant of which is of the form

$$\bar{F} = -\frac{\mu m}{r^2}\bar{r}^0 + mT\bar{r}^0 = \left(-\frac{\mu m}{r^2} + mT\right)\bar{r}^0.$$

The change in the kinetic energy of the craft is equal to the sum of these forces expended in moving it from the beginning of the liftoff (instant of engine firing) when $r_0 = R + H$ to the soft landing ($r = R$):

$$\frac{mv_{lan}^2}{2} - \frac{mv_0^2}{2} = \mu m\left(\frac{1}{R} - \frac{1}{r_0}\right) + mT(R - r_0).$$

By substituting $v_{lan} = 0$ in this equality and dividing both sides by m, we get the equation

$$-\frac{v_0^2}{2} = \frac{\mu(r_0 - R)}{Rr_0} + T(R - r_0). \quad /119$$

from which we can determine the altitude at which the engine is

fired, by solving the corresponding quadratic equation for r_0. The altitude is

$$H=\frac{1}{2T}\left[\frac{\mu}{R}+TR+\frac{v_0^2}{2}\pm\sqrt{\left(\frac{\mu}{R}+TR+\frac{v_0^2}{2}\right)^2-4\mu T}\right]-R. \qquad (11.1)$$

Determine with which sign we must take the root. By denoting the radicand with A, let us write (11.1) as $\frac{v_0^2}{2}\pm\sqrt{A}=$ $=2TH+TR-\frac{\mu}{R}$, and let us write the integral of energy in the form $\frac{v_0^2}{2}=TH-\frac{\mu}{R}+\frac{\mu}{r_0}$, so that $Tr_0-\frac{\mu}{r_0}=\pm\sqrt{A}$. From this it follows that when $T>\frac{\mu}{(R+H)^2}$, the root must be taken with the "plus" sign, and when $T<\frac{\mu}{(R+H)^2}$ -- with the "minus" sign. But since $\frac{v_0^2}{2}=\left[T-\frac{\mu}{R(R+H)}\right]H>0$, for a soft landing it is necessary that $T>\frac{\mu}{R(R+H)}>\frac{\mu}{(R+H)^2}$. This means that in Eq. (11.1) there must only be the "plus" sign in front of the root.

Problem 11.3. Set up the equations of rectilinear motion of a point with mass m in the Earth's gravity field for the cases of ascent from the Earth's surface ($r = R_{\oplus}$) to altitude H and falling to the Earth's surface from the same altitude without initial velocity. Determine the time of motion and calculate it for the case $H = R_{\oplus}$.

Solution. The equations of rectilinear motion in the Earth's gravity field are of the form

$$m\frac{d^2r}{dt^2}=-\frac{fmM_{\delta}}{r^2},\quad \frac{d^2r}{dt^2}=-\frac{fM_{\delta}}{r^2}=-\frac{\mu_{\delta}}{r^2}=-\frac{gR_{\delta}^2}{r^2}.$$

In problem 1.8 the launch velocity $v_0=\sqrt{\frac{2gR_{\delta}H}{R_{\delta}+H}}$, needed to bring a point to altitude H was determined. It is equal to the landing velocity of the point when falling from the same height, so that the initial equations are of the form:

for the ascent $t=0,\ r_0=R_{\delta},\ \dot{r}_0=v_0=\sqrt{\frac{2gR_{\delta}H}{R_{\delta}+H}}>0,$

for the descent $t=0,\ r_0=R_{\delta}+H,\ \dot{r}=0.$

In solving the second-order nonlinear equation $\ddot{r} = -\frac{gR_\delta^2}{r^2}$, we /120
must use the substitution $\ddot{r}\,dr = \dot{r}\,d\dot{r}$, after which the equation takes on the form $\dot{r}\,d\dot{r} = -\frac{gR_\delta^2}{r^2}\,dr$. Integrating it, we get the integral of energy $\frac{\dot{r}^2}{2} = \frac{gR_\delta^2}{r} + C$, where we find the constant $C = \left(\frac{\dot{r}^2}{2} + \frac{gR_\delta^2}{r}\right)_{t=0}$ from the initial conditions, so that in both cases $C = -\frac{gR_\delta^2}{R_\delta + H}$. Let us determine the time of motion from the integral of area, by transforming it into a first-order equation with separable variables:

$$dt = \pm\sqrt{\frac{r}{2(gR_\delta^2 + Cr)}}\,dr = \pm\frac{1}{R}\sqrt{\frac{R+H}{2g}}\sqrt{\frac{r}{R+H-r}}\,dr,$$

where the "plus" sign corresponds to the ascent (dr > 0), and the "minus" sign corresponds to the descent (dr < 0) of the point. Thus, the ascent time is equal to the descent time:

$$t = \frac{I}{R}\sqrt{\frac{R+H}{2g}}, \quad \text{where} \quad I = \int_R^{R+H}\sqrt{\frac{r}{R+H-r}}\,dr.$$

The integral I is taken by means of the substitution

$$\frac{r}{R+H} = \sin^2\varphi, \quad \frac{1}{R+H}\sqrt{\frac{r}{R+H-r}}\,dr = 2\sin^2\varphi\,d\varphi = (1-\cos 2\varphi)\,d\varphi,$$

so that

$$\frac{I}{R+H} = \int_{\varphi_0}^{\pi/2}(1-\cos 2\varphi)\,d\varphi = \frac{\pi}{2} - \varphi_0 + \sin\varphi_0\cos\varphi_0,$$

where

$$\sin\varphi_0 = \sqrt{\frac{R}{R+H}} = \cos\left(\frac{\pi}{2} - \varphi_0\right), \quad \sin\varphi_0\cos\varphi_0 = \frac{\sqrt{RH}}{R+H},$$

$$\frac{\pi}{2} - \varphi_0 = \arccos\sqrt{\frac{R}{R+H}} = \frac{1}{2}\arccos\frac{R-H}{R+H} = \arcsin\sqrt{\frac{H}{R+H}}.$$

Finally, we have

$$t = \frac{1}{R}\sqrt{\frac{R+H}{2g}}\left(\sqrt{RH} + \frac{R+H}{2}\arccos\frac{R-H}{R+H}\right), \qquad (11.2)$$

or

$$t = \frac{1}{R}\sqrt{\frac{R+H}{2g}}\left[\sqrt{RH} + (R+H)\arcsin\sqrt{\frac{H}{R+H}}\right]. \qquad (11.2')$$

We can easily see that Eq. (11.2'), as R → ∞, is transformed into /121 the formula of uniformly accelerated motion $t = \sqrt{2H/g}$, corresponding to Gallileo's formula $v = \sqrt{2gH}$ (see problem 1.8).

Let us calculate the time of motion of a point for the altitude $H = R_{\text{+}}$ by formula (11.2):

$$t = \sqrt{\frac{R_{\text{+}}}{g}}\left(1 + \frac{\pi}{2}\right) = 2{,}57\sqrt{\frac{6370\cdot 10^3}{9{,}8}} = 2065 \text{ sec} = 34.5 \text{ min}.$$

Problem 11.4. Two points with masses m and M begin to move from a rest state under the influence of forces of mutual attraction. Determine the time T by which the points will collide if the initial distance between them is L.

Solution. As indicated in problem 10.1, the relative motion of the points in the common two-body problem can be replaced by the absolute motion of one of them, executed under the influence of the gravity force of a new mass M + m, and thus we can reduce this problem to the problem of the motion of a moving point relative to a fixed point. The latter problem was already solved by us for the limited two-body problem (see problem 11.3). By substituting in Eq. (11.2), H = L, and $gR^2 = \mu = f(M + m)$, and by passing to the limit as R → ∞, we find

$$T = \frac{\pi}{2}\sqrt{\frac{L^3}{2\mu}}. \qquad (11.3)$$

We note that this result can be obtained directly from the integral of energy $\dot{r}^2 - \frac{2\mu}{r} = h$. Actually, for the case of descent $(\dot{r}_0 = 0,\ r_0 = L)$ when t = 0, we get $h = -\frac{2\mu}{L}$ and $\frac{dr}{dt} = -\sqrt{2\mu}\sqrt{\frac{1}{r} - \frac{1}{L}}$.

whence

$$T = -\frac{1}{\sqrt{2\mu}}\int_L^0 \sqrt{\frac{rL}{L-r}}\,dr = \frac{1}{\sqrt{2\mu}}\int_0^L \sqrt{\frac{rL}{L-r}}\,dr.$$

Making the substitution $r = L\sin^2\varphi$, we again get

$$T = \sqrt{\frac{L^3}{2\mu}}\int_0^{\pi/2} \sin^2\varphi\,d\varphi = \frac{\pi}{2}\sqrt{\frac{L^3}{2\mu}}.$$

The hypothetical time of fall of the point to Earth adopted as the attracting center with gravitational parameter $\mu_\oplus$ from altitude $H = L = R_\oplus$, calculated by this formulas, is $t = 890$ sec = = 14.8 min.

Problem 11.5. A heavy sphere moves along an imaginary rectilinear channel extending through the Earth's center. The force of attraction within the Earth is proportional to the distance to its center and is oriented toward this center. Determine the velocity at which the sphere transits the Earth's center and the velocity at which departs from the surface for the cases $v_0 = 0$ and $|v_0|>0$, and also the time of motion in both cases. /122

Solution. Writing the condition of equality of attractive force and gravity at the surface of the Earth, we get

$\bar{F} = -k^2 m\bar{r}$, $|\bar{F}|_{r=R_\delta} = k^2 mR_\delta = mg$, and $k^2 = g/R_\oplus$, on the basis of which the exact law of action of the force will be of the form

$\bar{F} = -\frac{mgr}{R_\delta}\bar{r}^0 = -\frac{mg}{R_\delta}\bar{r}$. The elementary work done by this force is $\delta A = \bar{F}\cdot d\bar{r} = -\frac{mg}{R_\delta}\bar{r}\,d\bar{r}$, while the total work done as the sphere moves within the globe is $A = -\frac{mg}{R_\delta}\times\int_{r_{beg}}^{r_{fi}} r\,dr = -\frac{mg}{2R_\delta}(r_{fi}^2 - r_{beg}^2)$.

Determine the change in the kinetic energy of the point as it moves toward the center and from the center toward the surface:

$$\frac{mv_c^2}{2} - \frac{mv_0^2}{2} = -\frac{mg}{2R_\delta}(0 - R_\delta^2) = \frac{1}{2}mgR_\delta > 0,$$

$$\frac{mv_1^2}{2} - \frac{mv_c^2}{2} = -\frac{mg}{2R_\delta}(R_\delta^2 - 0) = -\frac{1}{2}mgR_\delta < 0,$$

where v_1 is the velocity of departure at the surface, always equal to v_0. The velocity of transiting the center is $v_c = \sqrt{v_0^2 + gR_\delta} = \sqrt{v_1^2 + gR_\delta}$. For the case of motion without initial velocity, it is $v_c = \sqrt{gR_\delta} =$ $= 7.90$ km/sec. The equality $v_0 = v_1 = 0$ means that the sphere at once will fall back and its further motion will be an undamped harmonic oscillation. If $v_0 = v_1 > 0$, this motion will occur in the "channel" whose length would be apparently increased by 2H, where H is the altitude at which the sphere is lifted above the Earth. The law of oscillation in this case will be more complex, since the force of attraction $F = -\mu_\delta \bar{r}^0/r^2$ operates outside the Earth (see problem 11.3).

The equation of motion of the sphere within the Earth is a homogeneous second-order linear equation (the equation of free harmonic oscillation): $m\ddot{r} = -k^2 mr$, or $\ddot{r} + k^2 r = 0$, where $k = \sqrt{g/R_\delta}$. Its general solution, as we know, is of the form $r = C_1 \cos kt + C_2 \sin kt$, so that $\dot{r} = -C_1 k \sin kt + C_2 k \cos kt$. /123

Let us look at four cases of motion.

I. Falling to the center without initial velocity $(v_0 = v_1 = 0)$:

$$r_0 = R_\delta, \quad \dot{r}_0 = v_0 = 0, \quad C_1 = R_\delta, \quad C_2 = 0, \quad r_I = R_\delta \cos\sqrt{\frac{g}{R_\delta}}\, t,$$

$$t_I = \sqrt{\frac{R_\delta}{g}} \arccos 0 = \frac{\pi}{2}\sqrt{\frac{R_\delta}{g}}.$$ When $t = 0$ $r_I = R_\delta$, $t = t_I$, $r_I = 0$.

It is of interest to compare the calculated $t_I = 21.1$ min with the hypothetical time of fall of the point to Earth, attracted to the fixed center. It is, as was shown in problem 11.4, 14.8 min. This difference is due to the fact that for the law of attraction $F \sim 1/r^2$ the acceleration of a point with this same section of the trajectory will be greater than for the law of attraction $F \sim r$.

II. Falling to the Earth's center with initial velocity $|v_0| > 0$:

$$r_0 = R_\delta, \; \dot{r}_0 = v_0 = -|v_0| < 0, \; C_1 = R_\delta, \; C_2 = \frac{v_0}{k} = -|v_0|\sqrt{\frac{R_\delta}{g}} < 0,$$

$$r_{II} = R_\delta \cos\sqrt{\frac{g}{R_\delta}}\, t - |v_0|\sqrt{\frac{R_\delta}{g}} \sin\sqrt{\frac{g}{R_\delta}}\, t = \cos\sqrt{\frac{g}{R_\delta}}\, t \left(R_\delta - |v_0|\sqrt{\frac{R_\delta}{g}} \operatorname{tg}\sqrt{\frac{g}{R_\delta}}\, t\right),$$

$$t_{II} = \sqrt{\frac{R_\delta}{g}} \operatorname{arctg} \frac{\sqrt{gR_\delta}}{|v_0|}, \qquad t_{II}(v_0 = 0) = t_I.$$ When $t = 0$ $r_{II} = R_\delta$, $t = t_{II}$, $r_{II} = 0$.

III. Ascent to Earth's surface, where the sphere halts ($v_1 = 0$):

$$r_0 = 0, \quad \dot{r}_0 = v_c = \sqrt{gR_\delta} > 0, \quad C_1 = 0, \quad C_2 = \frac{v_c}{k} = \sqrt{gR_\delta}\sqrt{\frac{R_\delta}{g}},$$

$$r_{III} = R_\delta \sin\sqrt{\frac{g}{R_\delta}}\,t, \quad t_{III} = \sqrt{\frac{R_\delta}{g}} \arcsin 1 = \frac{\pi}{2}\sqrt{\frac{R_\delta}{g}} = t_I.$$

When $t = 0 \quad r_{III} = 0, \quad t = t_{III}, \quad r_{III} = R_\delta$. /124

IV. Ascent to the Earth's surface, where at the residual velocity $|v_1| > 0$:

$$r_0 = 0, \ \dot{r}_0 = v_c = \sqrt{gR_\delta + v_1^2}, \ C_1 = 0, \ C_2 = \frac{v_c}{k} = \sqrt{gR_\delta + v_1^2}\sqrt{\frac{R_\delta}{g}},$$

$$r_{IV} = \sqrt{gR_\delta + v_1^2}\sqrt{\frac{R_\delta}{g}} \sin\sqrt{\frac{g}{R_\delta}}\,t, \quad t_{IV} = \sqrt{\frac{R_\delta}{g}} \arcsin \frac{\sqrt{gR_\delta}}{\sqrt{gR_\delta + v_1^2}} =$$

$$= \sqrt{\frac{R_\delta}{g}} \operatorname{arctg} \frac{\sqrt{gR_\delta}}{v_1} = t_{II}, \quad t_{IV}(v_1 = 0) = \frac{\pi}{2}\sqrt{\frac{R_\delta}{g}} = t_{III} = t_I.$$

When $t = 0 \quad r_{IV} = 0, \quad t = t_{IV}, \quad r_{IV} = R_\delta$.

Finally, we have

$$t_I = t_{III}, \quad t_{II} = t_{IV}, \ t_I > t_{II}, \ t_{III} > t_{IV}.$$

Problem 11.6. A point with mass m moves under the influence of the central force $F = -m\left(\frac{\mu}{r^2} + \frac{\nu}{r^3}\right)$, where $\mu > 0$ and ν are certain constants. Determine the trajectory of the point, considering that the incremental force $- m\nu/r^3$ can be either a force of attraction or a force of repulsion (depending on the sign). Set up the equation of conservation of energy of the point, using Binet's formulas.

Solution. Based on Binet's second formula (see Chapter Four) in which $u = 1/r$ is the Binet variable and $c = r^2\dot{\phi}$ is the constant of areas, we can write

$$\frac{d^2u}{d\varphi^2} + u = -\frac{F}{mc^2u^2} = \left(\frac{\mu}{r^2} + \frac{\nu}{r^3}\right)\frac{1}{c^2u^2} = \frac{\mu}{c^2} + \frac{\nu u}{c^2},$$

from whence we obtain the inhomogeneous second-order linear equation in the Binet variable

$$\frac{d^2u}{d\varphi^2} + \left(1 - \frac{\nu}{c^2}\right)u = \frac{\mu}{c^2}, \tag{11.4}$$

from which when $\nu = 0$ there follows Eq. (4.11), given in problem 4.10. The solution of this equation can be represented in a different form, depending on the sign of the quantity $1 - \nu/c^2$. /125

I. For the case when $1 - \frac{\nu}{c^2} > 0$ $(\nu < c^2)$ we seek the solution of the form

$$u = \frac{\mu}{c^2\left(1 - \frac{\nu}{c^2}\right)} + A\cos\sqrt{1 - \frac{\nu}{c^2}}(\varphi - \varepsilon) = \frac{\mu}{c^2 - \nu}\left[1 + \frac{A(c^2 - \nu)}{\mu} \times \right.$$
$$\left. \times \cos\sqrt{1 - \frac{\nu}{c^2}}(\varphi - \varepsilon)\right],$$

from whence it is clear that the equation of the trajectory

$\left|\frac{\nu}{c^2}\right| \ll 1$ is the conic section

$$r = \frac{p}{1 + e\cos k(\varphi - \varepsilon)}, \tag{11.5}$$

that is moving, rotating about a focus, where $p = \frac{c^2 - \nu}{\mu}$, $k = \sqrt{1 - \frac{\nu}{c^2}}$, and $e = \frac{A(c^2 - \nu)}{\mu}$ (e and ε are arbitrary constants). Obviously, when $\nu = 0$, Eq. (11.5) is converted into the equation of the fixed conic section $r = \frac{p}{1 + e\cos(\varphi - \varepsilon)}$, obtained in problem 4.10 as the solution to Eq. (4.11) only for the attractive force $F \sim \frac{1}{r^2}$, that is, when motion is strictly periodic.

In this case, when the incremental force $F \sim 1/r^3$ is present, the motion of a point described by Eq. (11.5) loses its periodicity, that is, when ϕ is replaced with 2π and the radius-vector has an initial direction, its value differs from its initial value. Its initial value is restored when the angle ϕ is changed by the amount $\frac{2\pi}{k} = 2\pi\left(1 - \frac{\nu}{c^2}\right)^{-1/2}$, that is, when the radius-vector is rotated by an angle that is somewhat larger than 2π in

the case $\nu > 0$ (the incremental force is an attractive force), and by an angle somewhat smaller than 2π in the case $\nu < 0$ (the incremental force is a force of repulsion). When $e < 1$, the motion of the point in both cases is along an ellipse continuously rotating in its plane in the direction of motion of the point $\nu > 0$ or in the opposite direction, when $\nu < 0$ (positive or negative precession of orbit, where the angle of rotation of the focal axis of the precessing ellipse during the time of one complete revolution is $\alpha = 2\pi[(1 - \nu/c^2)^{-1/2} - 1]$. If ν/c^2 is sufficiently /126
small that its square can be neglected, we can write the approximate formula thusly:

$$\alpha \approx 2\pi\left[\left(1 + \frac{1}{2}\cdot\frac{\nu}{c^2}\right) - 1\right] = \frac{\pi\nu}{c^2}. \tag{11.6}$$

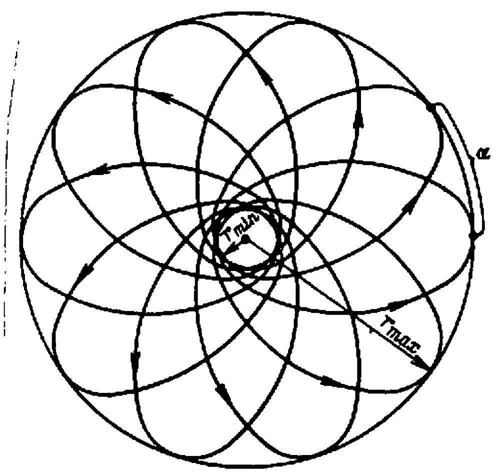

Fig. 30

The trajectory of motion has the form of a rosette (Fig. 30) continuously filling with lobes the circular part of the plane bounded by the circles

$$r_{min} = a(1-e) = \frac{p}{1+e}$$

and

$$r_{max} = a(1+e) = \frac{p}{1-e},$$

where p, α, and e are characteristics of the ellipse.

II. In the case when $1 - \frac{\nu}{c^2} = 0$ ($\nu = c^2$), Eq. (11.4) becomes

$$\frac{d^2u}{d\varphi^2} = \frac{\mu}{c^2} = \text{const}.$$

Its double integration in ϕ allows us to establish that the trajectory is a fairly complicated spirallike curve (precessing spiral) originating from the circle $r = r_0$ ($\phi = 0$).

III. In the case when $1-\frac{\nu}{c^2}<0$ ($\nu>c^2$), the trajectory is a spirallike curve with an infinite number of revolutions, originating from the circumscribed circle $r = r_{max}$ ($r \to 0$, $\phi \to \infty$).

IV. If $\mu = 0$, that is, if the force of attraction $F \sim 1/r^2$ disappears, Eq. (11.4) becomes uniform, and one of its solutions (when $\nu > c^2$) can be the equation of the logarithmic spiral $r = r_0\, e^{\lambda\phi}$. The solution of the equation of this type for the force $F \sim 1/r^3$ was obtained by us in problems 3.9 and 4.2.

The force fields of these types are encountered, for example, in the theory of motion of microparticles; however the case of /127
elliptical motion has an interesting interpretation also for planetary motions. To show this, let us set up the equation of conservation of energy of a point moving under the effect of the force indicated in the conditions of the problem, utilizing Binet's formulas.

Since the potential corresponding to the given force is of the form $U = m\,(\mu r^{-1} + 1/2\ \nu r^{-2})$, the law of conservation of energy $T-U = h$ becomes $\frac{1}{2} m v^2 - m\mu r^{-1} - \frac{1}{2} m\nu r^{-2} = h$,

whence we have

$$v^2 = \frac{2h}{m} + 2\mu u + \nu u^2.$$

Comparing this expression with Binet's first formula $v^2 = c^2\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right]$, we get the equation of energy

$$\left(\frac{du}{d\varphi}\right)^2 + u^2\left(1-\frac{\nu}{c^2}\right) - u\frac{2\mu}{c^2} = \frac{2h}{mc^2}. \tag{11.7}$$

This equation, defining the trajectory of a particle moving according to Newtonian mechanics under the combined influence of the forces $F \sim 1/r^2$ and $F \sim 1/r^3$ has an analogy in relativistic mechanics. It coincides in form with the equation of motion of a particle moving according to relativistic mechanics only under the influence of the force of attraction $F \sim 1/r^2$. Here we have in mind that the mass of a particle will vary in accordance with the Lorenz formula $m = m_0\left(1-\frac{v^2}{v_c^2}\right)^{-1/2}$, where m_0 is the rest mass, and v_c is the velocity of light.

By writing out the equation of conservation of energy, instead of the kinetic energy we must introduce the intrinsic energy of the particle $\varepsilon = m v_c^2$ playing the same role in relativistic mechanics as the kinetic energy in Newtonian mechanics.

Multiplying Binet's first formula by m^2, we get

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{m^2 v^2}{m^2 c^2}.$$

Let us set up for the numerator the formula derived from the Lorenz formula after identity transformations:

$$m^2 v^2 = m_0^2 v^2 \left(1 - \frac{v^2}{v_c^2}\right)^{-1} = \left(m_0^2 v^2 \frac{v_c^2}{v_c^2} \pm m_0^2 v_c^2\right) \times$$

$$\times \left(1 - \frac{v^2}{v_c^2}\right)^{-1} = m^2 v_c^2 - m_0^2 v_c^2.$$ /128

The quantity $m^2 v_c^2$ can be expressed in terms of the total energy of a particle: $E = \varepsilon - m\mu r^{-1} = m v_c^2 - m\mu u$,

so that $m^2 v_c^2 = v_c^{-2} (E + m\mu u)^2$ and $m^2 v^2 = v_c^{-2} (E + m\mu u)^2 - m_0^2 v_c^2$.

By substituting this expression into Binet's formula, we get the equation of energy

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 \left(1 - \frac{\mu^2}{c^2 v_c^2}\right) - u \frac{2E\mu}{mc^2 v_c^2} = \frac{E^2 - m_0^2 v_c^4}{m^2 c^2 v_c^2}, \tag{11.8}$$

coinciding in form with Eq. (11.6) if $m = m_0 = \text{const}$. Here the role of coefficient ν is played by μ^2/v_c^2. The coefficient μ can be replaced by $E\mu/m_0 v_c^2$, and so on.

This analogy of formulas enables us to give a qualitative and quantitative estimation of the relativistic effect of displacement of Mercury's perihelion.

The change in the mass of Mercury for motion along an orbit proves to be more significant than for other planets, since on the one hand its orbital velocity (mean) is higher than for these planets. On the other hand, its orbit has a fairly high eccen-

tricity e = 0.206 so that the velocity change in the motion from perihelion to aphelion also proves to be greater than for other planets. Because of this, finding the influence of the relativistic dependence of mass on velocity on the nature of the motion of Mercury, manifested in the precession of the orbit and the displacement of the perihelion, is easier than for other planets.

The analogy of equations found above enables us to use Eq. (11.6) in calculating the angle α of displacement of the perihelion under the influence of the velocity dependence of mass. After the coefficient ν has been replaced by $\mu_{\odot}^2/v_c^2$, Eq. (11.6) becomes

$$\alpha = \pi\mu_{\odot}^2/c^2 v_c^2 .$$

We know that $v_c = 3\cdot10^5$ km/sec. We can determine the constant of areas c for Mercury's orbit from these velocities and distances:

$r_\pi = 46\cdot10^6$ km, $v_\pi = 57.8$ km/sec, $r_\alpha = 70\cdot10^6$ km, $v_\alpha = 38.0$ km/sec,

so that $c = r_\pi v_\pi = r_\alpha v_\alpha = 2.66\cdot10^9$ km^2/sec^2. The displacement of the perihelion in one orbit of Mercury around the Sun (T = 88 days)

is $\alpha = \frac{3.14\,(1327\cdot10^8)^2}{(3\cdot10^5)^2\,(2.66\cdot10^9)^2} = 8.62\cdot10^{-8}$ rev^{-1}, which corresponds to the

secular displacement $\alpha = 7''.2$. This value of α is about one- /129
sixth of the observed displacement 42''.9 per century, whose value is due also to other relativistic effects.

This effect of the displacement of the planetary perihelion can be justified also from the standpoint of the law of conservation of the kinetic energy of a point in a central force field (law of areas). Actually, since the planetary velocity of the perihelion v_π is at a maximum, the planetary mass m_π is also larger than the mass at the aphelion m_α. Due to this law of areas, $mv_\pi r_\pi = mv_\alpha r_\alpha$ is not satisfied for a point with constant mass, and excess kinetic moment appears at the perihelion:

$$\Delta K = m_\pi v_\pi r_\pi - m_\alpha v_\alpha r_\alpha = v_\alpha r_\alpha (m_\pi - m_\alpha) = v_\pi r_\pi (m_\pi - m_\alpha) =$$

$$= m_0 v_\pi r_\pi \left[\left(1 - \frac{v_\pi^2}{v_c^2}\right)^{-1/2} - \left(1 - \frac{v_\alpha^2}{v_c^2}\right)^{-1/2}\right] \approx$$

$$\approx m_0 v_\pi r_\pi \left(1 + \frac{1}{2}\cdot\frac{v_\pi^2}{v_c^2} - 1 - \frac{1}{2}\cdot\frac{v_\alpha^2}{v_c^2}\right) = m_0 v_\pi r_\pi \frac{v_\pi^2 - v_\alpha^2}{2v_c^2} .$$

However, the orbital planetary motion is impossible without satisfying the law of areas. Therefore this excess moment at the perihelion ΔK is compensated by the incremental kinetic moment $\Delta K'$ arising due to the rotation of the orbit around the sun in the direction of planetary motion, that is, it is caused by the positive precession of the orbit. This incremental moment $\Delta K'$ will be larger at the aphelion than at the perihelion because $r_\alpha > r_\pi$. It is

$$\Delta K' = m_\alpha v_\alpha r_\alpha - m_\pi v_\pi r_\pi = (m_\alpha r_\alpha^2 - m_\pi r_\pi^2)\omega \approx m_0 \omega (r_\alpha^2 - r_\pi^2),$$

where ω is the angular velocity of orbital revolution satisfying the condition $\omega = \frac{v_\alpha}{r_\alpha} = \frac{v_\pi}{r_\pi}$. By equating ΔK to $\Delta K'$, we get

$$\omega = \frac{v_\pi r_\pi}{2 v_c^2} \cdot \frac{v_\pi^2 - v_\alpha^2}{r_\alpha^2 - r_\pi^2} = \frac{57.8 \cdot 46 \cdot 10^6}{2(3 \cdot 10^5)^2} \cdot \frac{(57.8)^2 - (38.0)^2}{(70 \cdot 10^6)^2 - (46 \cdot 10^6)^2} =$$

$$= 1.02 \cdot 10^{-14} \text{ sec}^{-1}.$$

Since a century contains $t = 3.15 \cdot 10^9$ sec, the rotation of Mercury's orbit in a century in arc seconds (1 rad = $2.06 \cdot 10^5{}''$) /130 is $\alpha'' = \omega'' t = (1.02 \cdot 10^{-14})(2.06 \cdot 10^5)(3.15 \cdot 10^9) \approx 7''.2$.

Problem 11.7. A spacecraft with mass m moves in a central attractive field under the influence of the force $\bar{F} = -\frac{\mu m}{r^2} \bar{r}^0$ in a circular orbit with radius r_0. On the craft acts the engine thrust $\bar{T} = \frac{\alpha \mu m}{r^2} \bar{r}^0$ (a continuously-acting variable radial thrust). Determine the nature of the change in the craft's orbit as a function of the change in the magnitude and sign of the constant coefficient α. (Problem of flight with radial thrust.)

Solution. After firing the engine, the craft is continuously under the influence of two forces directed along the same straight line (along the radius-vector), where when $\alpha < 0$ both forces are directed toward the force center, and when $\alpha > 0$ -- to opposite sides. The resultant of these forces is

$$\bar{F}' = \bar{F} + \bar{T} = -\frac{(1-\alpha)\mu m}{r^2} \bar{r}^0 = -\frac{\mu' m}{r^2} \bar{r}^0.$$

The replacement of $(1 - \alpha)\mu$ by the new gravitational constant μ' signifies "replacing" the true gravitational μ-field by the imaginary μ'-field. Also valid for the new field are Kepler's laws, since no fundamental changes occur in the equations of motion. This replacement can be interpreted as a "change" in the mass of the attracting center by the amount αM so that the relation of the gravitational parameters is as follows:

$$\mu = fM, \quad \mu' = (1-\alpha)\mu = (1-\alpha)fM = f(M - \alpha M) = fM'.$$

Let us look at several particular cases.

I. $\alpha < 0$ (radial thrust is directed toward the attracting center). At the instant of engine firing the μ-field is replaced by the μ'-field, where $\mu' = (1 - \alpha)\mu > \mu$. The motion of the craft along the former circular orbit with circular velocity $v = \sqrt{\mu/r_0}$ now proves to be impossible since the theoretical value of the circular velocity for a given r_0 in the μ'-field is $v' = \sqrt{\mu'/r_0} > 0$. In other words, the actual velocity $v < v'$ proves to be elliptical with respect to the μ'-field and the craft passes into an elliptical orbit situated within the initial circular orbit (Fig. 31), where the engine firing point M_0 proves to be the orbital apocenter.

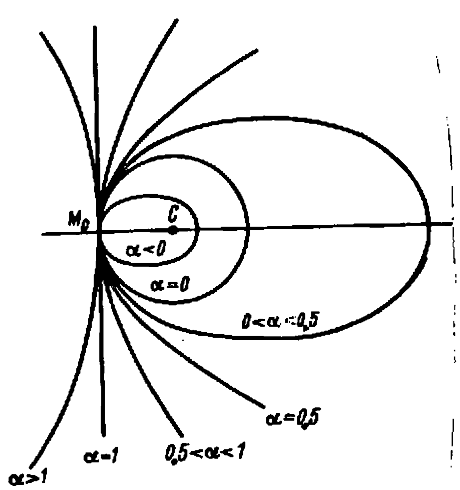

Fig. 31

II. $\alpha = 0$ (thrust is absent). /131 In this case no changes in craft motion occur.

III. $0 < \alpha < 0.5$ (the radial thrust is directed from the center). At the moment the μ-field is replaced by the μ'-field, when $\mu' = (1 - \alpha)\mu < \mu$, the actual circular velocity $v = \sqrt{\mu/r_0}$ proves to be larger than the theoretical circular velocity $v' = \sqrt{\mu'/r_0}$ and is elliptical ($v > v'$) with respect to the μ'-field. The craft passes into an elliptical orbit lying outside the circular orbit, and the point M_0 becomes its pericenter.

IV. $\alpha = 0.5$. In this case the actual velocity $v = \sqrt{\mu/r_0}$ is parabolic with respect to the μ'-field, since the theoretical value of the circular velocity $v' = \sqrt{\frac{\mu'}{r_0}} = \sqrt{\frac{0.5\mu}{r_0}} = \frac{v}{\sqrt{2}}$, whence $v = v'\sqrt{2}$, and the craft leaves the μ'-field along a parabola.

V. $\alpha = I$ (one field "damps" another). The gravitational field is entirely eliminated and the craft moves rectilinearly

and uniformly with velocity $v > 0$ in a gravity-free space, along a tangent to the initial circular orbit. The circular velocity with respect to the μ'-field here is absent: $v' = \sqrt{\frac{(1-\alpha)\mu}{r_0}} = 0$.

When $0.5 < \alpha < 1$, we get hyperbolic orbits with respect to the μ'-field ($v > v'\sqrt{2}$).

VI. $\alpha > 1$. The effect of superimposing the fields is such as to give the effect of the central field repelling the craft. The circular velocity v' in the μ'-field does not exist at all (it is imaginary). The second branch of a hyperbola not containing the focus O at which the central mass is situated is the trajectory of motion.

We note that the equivalent of forces $\bar{F}' = \bar{F} + \bar{T}$, just like the force $\bar{F}$, is a potential force, as the result of which the actual radial thrust $\bar{T}$ is a potential force:

$$U_F = \frac{\mu m}{r}, \quad U_{F'} = \frac{\mu' m}{r} = \frac{(1-\alpha)\mu m}{r} = \frac{\mu m}{r} - \frac{\alpha \mu m}{r}, \quad U_{\bar{T}} = -\frac{\alpha \mu m}{r}.$$ /132

The thrust of a plane mirror solar sail positioned perpendicular to the solar rays can serve as an example of a radial thrust varying under the law $T \sim 1/r^2$, since the force of solar radiation pressure varying according to the law $F \sim 1/r^2$ can be viewed as a thrust force (we have in mind heliocentric motion).

Problem 11.8. Prove that the motion of a spacecraft can follow a logarithmic spiral $r = r_0 e^{\lambda\phi}$ if, in addition to the attractive force of the central body $F \sim 1/r^2$, the craft is continuously acted on by a variable thrust $R \sim 1/r^2$ tangent to the trajectory. Determine the velocity along the spiral. Give the energy characteristic of the thrust R. Find the time of motion along the spiral. (Problem with tangential thrust.)

Solution. Since the point moves in a logarithmic spiral $r = r_0 e^{\lambda\phi}$ ($\lambda = \mathrm{ctg}\,\beta$) (Fig. 32) under the effect of a single central force (see problem 4.2), this problem can only be the force $F \sim 1/r^3$ (see also problems 3.9 and 11.6). However, it can be shown that this motion is possible under the influence of the attractive force $F \sim 1/r^2$ (force of Newtonian attraction) if to it we add the tangential thrust $F \sim 1/r^2$. Let us write out the equations of motion in polar coordinates: $m(\ddot{r} - r\dot{\varphi}^2) = F_r + R_r$, and $m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = R_\varphi$,

whence the radial and transversal craft accelerations are as follows:

$$w_r = \ddot{r} - r\dot{\varphi}^2 = -\frac{\mu}{r^2} + \frac{R}{m}\cos\beta, \quad w_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = \frac{R}{m}\sin\beta. \tag{11.9}$$

By differentiating the equation of the spiral, we find /133
$\dot{r} = r\dot{\varphi}\,\mathrm{ctg}\,\beta$ and $\ddot{r} = r\dot{\varphi}^2\,\mathrm{ctg}^2\beta + r\ddot{\varphi}\,\mathrm{ctg}\,\beta$ From the second equation of (11.9) we have $\frac{R}{m} = \frac{1}{\sin\beta}(r\ddot{\varphi} + 2r\dot{\varphi}^2\,\mathrm{ctg}\,\beta)$. By substituting this ratio into the first equation of (11.9) we get:

$$r\dot{\varphi}^2(1 + \mathrm{ctg}^2\beta) = \frac{\mu}{r^2}, \quad \dot{\varphi}^2 = \frac{\mu}{r^3}\sin^2\beta,$$

whence

$$2\dot{\varphi}\ddot{\varphi} = -\frac{3\mu\dot{r}}{r^4}\sin^2\beta, \quad \ddot{\varphi} = -\frac{3}{2}\dot{\varphi}^2\,\mathrm{ctg}\,\beta = -\frac{3}{2}\frac{\mu}{r^3}\sin\beta\cos\beta.$$

If from the first equation of (11.9) we determine R/m and replace the derivatives $\ddot{r}$, $\dot{\varphi}^2$ and $\ddot{\varphi}$ by their expressions, we find the acceleration imparted to the craft by the tangential thrust, therefore, we find the thrust R:

$$\frac{R}{m} = \frac{1}{\cos\beta}\left(\ddot{r} - r\dot{\varphi}^2 + \frac{\mu}{r^2}\right) = \frac{\mu}{r^2}\left(1 - \sin^2\beta + \cos^2\beta - \frac{3}{2}\cos^2\beta\right) = \frac{\mu\cos\beta}{2r^2},$$
$$R = \mu m\cos\beta/2r^2.$$

Thus, for the motion to occur along a logarithmic spiral, the tangential thrust $\bar{R}$ must be a force of repulsion proportional to $1/r^2$. These conditions are satisfied, for example, by the pressure of solar radiation so that the theoretical possibility of a craft with a solar sail moving in outer space is evident when the spiral angle β is equal to the angle of sail positioning, that is, equal to the angle of incidence of the radiation at the sail surface γ. Taking the motion to be heliocentric ($\mu = \mu_\odot$), let us determine the craft velocity along a spiral, by replacing the derivative $\dot{r}$ in the formulas by the radial and transversal velocities:

$$v_r = \dot{r} = r\dot{\varphi}\,\mathrm{ctg}\,\beta = \sqrt{\frac{\mu_\odot}{r}}\cos\beta, \quad v_\varphi = r\dot{\varphi} = \sqrt{\frac{\mu_\odot}{r}}\sin\beta,$$

so that

$$v = \sqrt{v_r^2 + v_\varphi^2} = \sqrt{\frac{\mu_\odot}{r}}. \tag{11.10}$$

This means that at each point of the spiral the craft velocity, always oriented along the tangent to this spiral, is equal to the local circular heliocentric velocity at the given point and decreases with growing separation of the craft from the Sun. Using this formula we can get the energy characteristics of the thrust $\bar{R}$ by writing the change in the craft's kinetic energy as it passes from one point of the spiral to the other as equal to the sum of work done by the solar gravity and the work done by the thrust: /134

$$\frac{mv_2^2}{2} - \frac{mv_1^2}{2} = A_{F_\odot} + A_R,$$

or

$$\frac{m\mu_\odot}{2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = m\mu_\odot\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + A_{\bar{R}},$$

so that the work done by the thrust $A_R = -\frac{1}{2} m\mu_\odot\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$ is positive as the craft goes farther and farther from the Sun ($r_2 > r_1$) (in contrast to the work done by the gravity) and is negative with increasing proximity to the Sun ($r_2 < r_1$). In the case when the craft goes away from the Sun, the kinetic energy drops off. Just like the velocity, the work done by the thrust $A_{\bar{R}}$ does not depend on the shape of the spiral (on the spiral angle β), that is, does not depend on the number of orbits that the craft must make around the Sun to pass from one point to another.

We can easily see that the tangential thrust $\bar{R}$ is not a potential force. We can be convinced of this by writing out the projections of the resultant $\bar{F}' = \bar{F} + \bar{R}$:

$$F_r' = -\frac{\mu_\odot m}{r^2} + \frac{\mu_\odot m\cos^2\beta}{2r^2} = -\frac{\mu_\odot m}{r^2}\left(1 - \frac{1}{2}\cos^2\beta\right),$$

$$F_\varphi' = \frac{\mu_\odot m}{2r^2}\cos\beta \sin\beta .$$

Actually, the force $\bar{F}'$ is not a potential force since it is impossible to select the force function U' satisfying simultaneously the conditions of potentiality $F_r' = \frac{\partial U'}{\partial r}$ and $F_\varphi' = \frac{1}{r}\cdot\frac{\partial U}{\partial \varphi}$. So the force $\bar{R}$ is not a potential force as well. We note that the resultant $\bar{F}'$ is also not a central force, as the result of which neither the integral of areas nor the Binet formulas are satisfied in this problem.

To determine the time of flight along the spiral it is sufficient to integrate any of the relations for the velocity projections, for example, $\frac{d\varphi}{dt} = \sqrt{\frac{\mu_0}{r^3}} \sin\beta$. Introducing the value of r from the equation of a spiral into this relation, we get the equation $e^{3\lambda\varphi/2}\, d\varphi = \sqrt{\frac{\mu_0}{r_0^3}} \sin\beta\, dt$, from whence we have

$$t = \frac{2\,(r_0 e^{\lambda\varphi})^{3/2}}{3\lambda\sqrt{\mu_0}\sin\beta} = \frac{2\,r^{3/2}}{3\sqrt{\mu_0}\cos\beta} .$$ /135

Thus, to pass from r_1 to r_2 takes the time

$$t = \frac{2}{3}\cdot\frac{(r_2^{3/2} - r_1^{3/2})}{\sqrt{\mu_0}\cos\beta} , \tag{11.11}$$

which depends on the spiral angle β $(0 \leqslant \beta \leqslant \frac{\pi}{2})$. If the angle β is small, the spiral is weakly generated, and the time of motion is short, where the minimum possible time $t = \frac{2\,(r_2^{3/2} - r_1^{3/2})}{3\sqrt{\mu_0}}$ corresponds to the case of radial motion $\beta = 0$ (see problem 11.7). If the angle β is large, then the spiral turns sharply and the time of motion is long. When $\beta = \pi/2$, we again get the initial circular orbit for which it is not possible to determine the time of motion based on Eq. (11.11). In this case we must use the formulas in Chapter Six.

We should note that the explicit or implicit (in terms of r) dependence of the motion characteristics on the spiral angle β is determined by their dependence on the parameters of the sail producing the thrust, since in this case we can assume that each β corresponds to fixed sail parameters. For example, we assume that some given "sail factor" of the craft corresponds to $\beta = 88°$ ($\cos\beta = 0.03$) (strict correspondence is possible only for the case when the spiral angle β is equal to the angle γ of incidence of the rays at the plane surface). Based on Eq. (11.11) we can calculate the time of motion of the craft from Earth orbit to Mars orbit:

$$t = \frac{2[(2.28\cdot10^8)^{3/2} - (1.50\cdot10^8)^{3/2}]}{3\sqrt{1327\cdot10^8}\cdot 0.03} = 0.98\cdot10^8 \text{ sec} = 1130 \text{ days} = 3.1 \text{ years}.$$

Here the radius-vector of the craft (heliocentric) will turn by the angle

$$\varphi = \frac{2}{3 \operatorname{ctg} \beta} \left| \ln \frac{3t\sqrt{\mu_{\odot}} \cos\beta}{2r^{3/2}} \right| =$$

$$= \frac{2}{3 \cdot 0.03} \left| \ln \frac{3 \cdot 0.98 \cdot 10^{8} \cdot \sqrt{1327 \cdot 10^{8}} \cdot 0.03}{2(1.50 \cdot 10^{8})^{3/2}} \right| = 3.12 = 178^{\circ},$$

that is, in the time of motion from the Earth orbit to Mars orbit the craft will make about half a revolution along the spiral around the Sun. /136

Problem 11.9. Determine the distance r from Earth's center to points situated along the same line as the centers of the Earth and Moon at which the force of attraction by the Earth of a mass m is equal to the force of attraction of this same mass by the Moon. Take the Earth-Moon distance to be d = 384,400 km, and the mass ratio $M_{☾}/M_{\oplus} = 1:81.5$. (Problem of sphere of attraction.)

Solution. The equality of forces of attraction of mass m by the Earth and by the Moon

$$|\bar{F}_{\oplus}| = |\bar{F}_{☾}| = \frac{fmM_{\oplus}}{r^2} = \frac{fmM_{☾}}{(d-r)^2}, \quad \frac{M_{\oplus}}{r^2} = \frac{M_{☾}}{(d-r)^2}$$

enables us to write out the quadratic equation that has the following roots:

$$(M_{\oplus} - M_{☾}) r^2 - 2M_{\oplus} rd + M_{\oplus} d^2 = 0,$$

$$r_{1,2} = \frac{d}{M_{\oplus} - M_{☾}} (M_{\oplus} \pm \sqrt{M_{\oplus} M_{☾}}) = \frac{d}{1 \mp \sqrt{M_{☾}/M_{\oplus}}}.$$

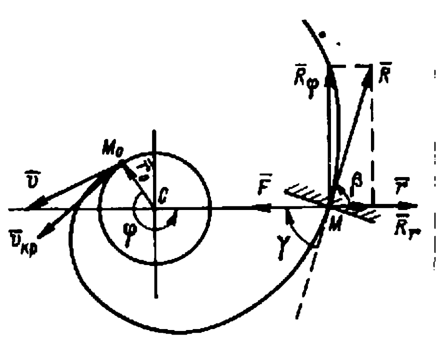

Fig. 32

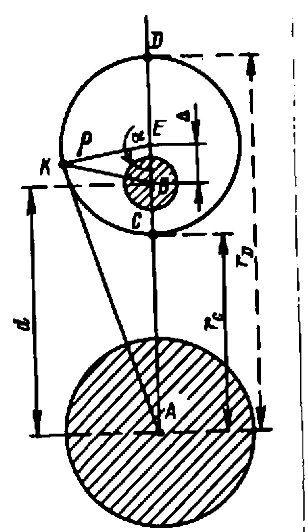

Fig. 33

Thus, there are at least two equi-attracted points situated along the lines of centers of attraction (collinear points) for which (Fig. 33) we have

$$r_C = \frac{d}{1+\sqrt{M_☾/M_♁}} \approx 0.90d,$$

$$r_D = \frac{d}{1-\sqrt{M_☾/M_♁}} \approx 1.12d,$$

so that the point C is between Earth and Moon at a distance r_C = = 346,000 km from the Earth's center and d - r_C = 38,400 km from the Moon's center, while point D lies "beyond the Moon" at the distance r_D = 430,500 km from Earth's center and r_D - d = 46,100 km from Moon's center. Point D is 7700 km farther from the Moon's center than is the point C.

A mass m placed at point C experiences equal attraction by both Earth and Moon, where these forces are directed in different /137 directions so that the mass, devoid of velocity, stays at point C. A mass placed at point D will also experience equal attraction, but the forces in this case are directed in the same direction so that the mass, devoid of velocity, will fall toward the Moon under the influence of the resultant of these attractive forces. The force of attraction at point D is smaller than at point C:

$$\frac{|\bar{F}_C|}{|\bar{F}_D|} = \frac{r_D^2}{r_C^2} = \left(\frac{1+\sqrt{M_☾/M_♁}}{1-\sqrt{M_☾/M_♁}}\right)^2 = 1.56.$$

The conditions of pairwise equality of the attractive forces of Earth and Moon enable us to write

$$\frac{BC}{AC} = \frac{d-r_C}{r_C} = \sqrt{\frac{M_☾}{M_♁}}, \quad \frac{BD}{AD} = \frac{r_D-d}{r_D} = \sqrt{\frac{M_☾}{M_♁}},$$

from whence it follows that points C and D divide up the distance between Earth's center and Moon's center AB in the ratio $\sqrt{M_☾/M_♁}$ internally and externally. Using the techniques of elementary geometry, we can prove that if a sphere is constructed on the segment CD as a diameter, the sphere will be the geometrical locus of points for each of which the force of Earth attraction is equal to the force of Moon attraction. The value of this force differs from the value at the neighboring force owing to the change in distance (for the same mass m). We call this sphere the sphere of attraction of the Moon relative to the Earth. Within this

sphere the force of attraction of mass m by the Moon is always greater than the force of attraction by the Earth. The radius of the sphere of attraction ρ and the distance of its center E from the Moon's center BE = Δ can be calculated using the formulas

$$\rho = \frac{1}{2} CD = \frac{1}{2}(r_D - r_C) = 42{,}300 \text{ km}, \qquad \Delta = r_C + \rho - d = 4740 \text{ km},$$

from which it follows that the center of the sphere of attraction lies "beyond the Moon" by a distance of 3100 km from its surface ($R_☾$ = 1740 km).

The concept of the "sphere of attraction of a lesser mass relative to a greater" applies to any point masses. To determine the parameters of the sphere if we know only the masses and the distances between the masses, we can use the following formulas that are valid for the Earth-Moon case:

$$r_C = d - (\rho - \Delta), \quad r_D = d + (\rho + \Delta),$$

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$$d - \rho + \Delta = \frac{d}{1 + \sqrt{M_☾/M_♁}}, \quad d + \rho + \Delta = \frac{d}{1 - \sqrt{M_☾/M_♁}},$$

$$\Delta = \frac{d M_☾/M_♁}{1 - M_☾/M_♁} = 7470 \text{ km}, \quad \rho = \frac{d\sqrt{M_☾/M_♁}}{1 - M_☾/M_♁}. \qquad (11.12)$$

By writing out these formulas for the Sun-Earth case, we can determine the parameters of the sphere of attraction of the Earth relative to the Sun: $M_♁ = 6 \cdot 10^{27}$ Gs, $M_\odot = 2 \cdot 10^{33}$ Gs, $d = 150 \cdot 10^6$ km, ρ = 259,500 km, and Δ = 450 km. Thus, the center of the Earth's sphere of attraction relative to the Sun lies within the globe. It is interesting to note that the Moon is outside the Earth's sphere of attraction relative to the Sun, as a consequence of which the Sun attracts the Moon more strongly than does the Earth. Actually, let us set up the ratio of the magnitude of the attractive forces acting on the Moon:

$$|\bar{F}_\odot| : |\bar{F}_♁| = (M_\odot/M_♁)(r_{♁☾}/r_{\odot☾})^2.$$

By adopting the mean distance from Moon to Sun as $150 \cdot 10^6$ km, we get

$$|\bar{F}_\odot| : |\bar{F}_♁| = \frac{2 \cdot 10^{33}}{6 \cdot 10^{27}} \left(\frac{3.84 \cdot 10^5}{1.50 \cdot 10^8}\right)^2 = 2.18,$$

so that the Sun attracts the Moon roughly twice as strongly as does the Earth. The stable geocentric motion of the Moon is due to the corresponding geocentric velocities and accelerations.

Using Eqs. (11.12) we can also show that any point K of a sphere constructed on diameter CD exhibits the above-indicated property of equality of attractive forces. To see this, it is sufficient to prove the following equality (see Fig. 33):

$$\frac{KB}{KA} = \text{const} = \frac{BC}{AC} = \sqrt{\frac{M_☾}{M_☉}} = \alpha .$$

By denoting $\alpha^2 = M_☾ / M_☉$, we can rewrite Eqs. (11.12) as

$$\Delta = \frac{d\alpha^2}{1-\alpha^2}, \quad \rho = \frac{d\alpha}{1-\alpha^2}, \quad d + \Delta = d + \frac{\alpha^2 d}{1-\alpha^2} = \frac{d}{1-\alpha^2} .$$

On the other hand, based on the theorem of cosines for the triangles KBE and AKE, we have

$$(KB)^2 = \Delta^2 + \rho^2 - 2\rho\Delta\cos\alpha = \left(\frac{d}{1-\alpha^2}\right)^2 (\alpha^4 + \alpha^2 - 2\alpha^3 \cos\alpha),$$ /139

$$(KA)^2 = (d+\Delta)^2 + \rho^2 - 2(d+\Delta)\rho\cos\alpha = \left(\frac{d}{1-\alpha^2}\right)^2 (1 + \alpha^2 - 2\alpha\cos\alpha),$$

from whence there follows $(KB/KA)^2 = \alpha^2$, that is, $KB/KA = \alpha$, which is what we set out to prove.

Problem 11.10. Determine at which velocity a missile must be launched from the Earth's surface aimed at the Moon for it to reach the point of equal attraction closest to the Earth and to remain in equilibrium at it.

Solution. As indicated in problem 11.9, the collinear point of equal attractions C situated along the line of the centers of the Earth and Moon (see Fig. 33) is the point of equal attractions closest to the Earth. The distance from the Earth's center to this point r_c = 346,000 km, and from the Moon's center -- $d - r_c$ = = 38,400 km. In this problem was also shown that if some mass (a missile or a spacecraft) reaches point C with zero velocity relative to Earth and Moon, this mass will remain at this point in equilibrium by virtue of the above presented equality of attractive forces.

When a missile moves from the Earth's surface to point C, a change occurs in its kinetic energy equal to the sum of work done

by the attractive forces of Earth and Moon:

$$\frac{mv_c^2}{2} - \frac{mv_0^2}{2} = A_{\bar{F}_{\oplus}} + A_{\bar{F}_{☾}},$$

where

$$A_{\bar{F}_{\oplus}} = fmM_{\oplus}\left(\frac{1}{r_c} - \frac{1}{R_{\oplus}}\right) < 0,$$

$$A_{\bar{F}_{☾}} = fmM_{☾}\left(\frac{1}{d - r_c} - \frac{1}{d - R_{\oplus}}\right) > 0.$$

The equality of the missile at point C means that it arrives there at zero velocity, so that the launch velocity is

$$v_0 = \sqrt{2fM_{\oplus}\left(\frac{1}{R_{\oplus}} - \frac{1}{r_c} - \frac{M_{☾}/M_{\oplus}}{d - r_c} + \frac{M_{☾}/M_{\oplus}}{d - R_{\oplus}}\right)}.$$

Thus, knowing that $fM_{\oplus} = 398{,}600$ km^3/sec^2, $R_{\oplus} = 6370$ km, /140
$d = 384{,}400$ km, $r_c = 346{,}000$ km, and $d - r_c = 38{,}400$ km, let us find $v_0 = 11.04$ km/sec.

Note that the formula $g = fM_{\oplus}/R_{\oplus}^2$ is no longer valid for the motion of a point in the field of attraction of two centers. Assuming the resultant force of attraction of Earth and Moon at the surface of the Earth to be equal to the weight of a body, we get

$$-mg\bar{r}^0 = fm\left[-\frac{M_{\oplus}}{R_{\oplus}^2} + \frac{M_{☾}}{(d - R_{\oplus})^2}\right]\bar{r}^0,$$

whence

$$g = f\left[\frac{M_{\oplus}}{R_{\oplus}^2} - \frac{M_{☾}}{(d - R_{\oplus})^2}\right].$$

When $M_{☾} = 0$, this formula is converted into the formula $gR_{\oplus}^2 = fM_{\oplus}$, which is valid for one attractive center.

Similarly, we determine the falling velocity of mass m onto the Moon if this mass is at the collinear point of equivalent attractions D without initial velocity. In this case

$$\frac{mv^2}{2} - \frac{mv_D^2}{2} = A_{\bar{F}_\delta} + A_{\bar{F}_\zeta},$$

$$A_{\bar{F}_\delta} = fmM_\delta\left(\frac{1}{d+R_\zeta} - \frac{1}{r_D}\right) > 0, \quad A_{\bar{F}_\zeta} = fmM_\zeta\left(\frac{1}{R_\zeta} - \frac{1}{r_D - d}\right) > 0,$$

so that

$$v = \sqrt{2fM_\delta\left(\frac{1}{d+R_\delta} - \frac{1}{r_D} + \frac{M_\zeta/M_\delta}{R_\zeta} - \frac{M_\zeta/M_\delta}{r_D - d}\right)}.$$

When R_ζ = 1740 km, r_D = 430,500 km, and d = 384,400 km, we get v = 2.38 km/sec. Obviously, this then must be the launch velocity from the surface of the Moon required for the rocket to reach the collinear point of equal attractions D, after which it falls to the surface of the Moon.

Problem 11.11. A spacecraft moves along an elliptical trajectory around an attractive center M with gravitational parameter μ_M. The craft passes near some attractive mass m. Assuming that the perturbing mass is in the orbital plane of the craft, determine the instantaneous change in the constant of energy $d\tilde{h}/dt$ and the semi-major axis $d\alpha/dt$ of the craft orbit.

Solution. The orbital energy of a craft moving in the field of attraction of the central mass M is determined by the integral of energy $\tilde{h} = v^2 - \frac{2\mu_M}{r} = -\frac{\mu_M}{\alpha}$. We will call the variation in $\tilde{h}$ and α for constant r the instantaneous change in these quantities: /141

$$\frac{d\tilde{h}}{dt} = 2v\frac{dv}{dt}, \quad \frac{d\alpha}{dt} = \frac{2\alpha^2 v}{\mu_M}\cdot\frac{dv}{dt},$$

where dv/dt is the instantaneous velocity change.

In this problem (see Fig. 34) this velocity change is defined thusly. Suppose the force $|\bar{F}_m| = \mu_M/\rho^2$ is a force of attraction of the perturbing mass m (m << M) acting on a unit-mass craft at the distance ρ from it. Under the influence of this force

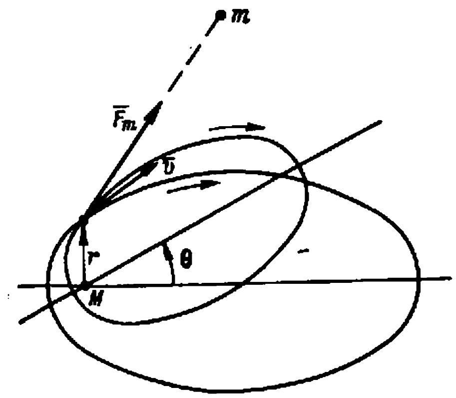

Fig. 34

tangential and normal craft accelerations are induced

$$w_\tau = \frac{\mu_m}{\rho^2}\cos\theta = \frac{dv}{dt},$$

$$w_n = \frac{\mu_m}{\rho^2}\sin\theta.$$

Of these accelerations, only the tangential acceleration w_τ can alter the orbital energy, so that

$$\frac{d\tilde{h}}{dt} = \frac{2\mu_m v}{\rho^2}\cos\theta, \quad \frac{da}{dt} = \frac{2a^2 v}{\rho^2}\cdot\frac{\mu_m}{\mu_*}\cos\theta.$$

From the resulting formulas it is clear that the orbital energy h increases or decreases, depending on the direction of craft motion, that is, on the change in the angle θ. The change in energy $d\tilde{h}/dt$ depends only on the gravitational parameter of the perturbing mass μ_m and the distance ρ at which the craft is separated from it. The value of cos θ must be assumed negative if the perturbing force causes orbital rotation in a direction opposite to the direction of the orbital motion of the craft. In this case the amount of energy ($\tilde{h} < 0$) and the semi-major axis are reduced. It is precisely this case that is shown in Fig. 34. /142

Problem 11.12. When an artificial Earth satellite is moving in the atmosphere, scattering (dissipation) of the total energy of the AES is observed, obeying the law $dh/dt = -2\Phi$, where Φ is the positive scattering function. Determine the nature of the evolution of the elliptical orbit of the AES $(h = \frac{m\tilde{h}}{2} < 0)$. Write out the equation of the energy balance and establish the nature of the change in the orbital kinetic and potential energies of the AES.

Solution. Based on the law of scattering, the total energy h acted on by atmospheric drag decreases. From the formula $h = -\mu m/2a$ it becomes obvious that the decrease in the negative h (elliptical orbit) is accompanied by a decrease in the total semi-major axis of the orbit a, that is, the orbit sinks into the denser atmospheric layers.

We can easily show that during this descent of the orbit, the apogee sinks faster than the perigee. Actually, let us look at one revolution of the orbit: apogee – perigee – apogee. At

the perigee the atmospheric density for a given revolution is the maximum, therefore the atmospheric drag appearing in the drag formula will also be at a maximum. In addition, the drag is a function of AES velocity, and its velocity at the perigee is at a maximum so that it is obvious that the effect of atmospheric drag will be at a maximum precisely at the perigee. As a result, the AES partially loses velocity at the perigee, resulting in the former apogee altitude no longer being attained. A similar effect can be observed for any point on the orbit, but its intensity decreases as the satellite moves from the perigee to the apogee (the density decreases, since the altitude of the AES increases, while the velocity diminishes according to the law of areas). At the apogee the deceleration effect will be the smallest, while a slight loss in velocity will not substantially influence the position of the next perigee. Thus, the apogee sinks faster than the perigee, resulting in the elliptical orbit being converted into a circular orbit with simultaneous reduction in α and r: $e \to 0$, and $\alpha \to r_\alpha \to r_\pi \to r$.

In spite of the loss of orbital velocity in each revolution, overall as the orbit sinks from revolution to revolution the mean /143
orbital velocity will increase. Actually, let us determine the limit of the mean AES velocity (see problem 5.19):

$$\lim_{a \to r} v_{av} = \frac{1}{2} \lim_{a \to r} (v_\pi + v_\alpha) = \frac{1}{2} \lim \left[\sqrt{\mu_\delta \left(\frac{2}{r_\pi} - \frac{1}{a} \right)} + \right.$$

$$\left. + \sqrt{\mu_\delta \left(\frac{2}{r_\alpha} - \frac{1}{a} \right)} \right] = \sqrt{\frac{\mu_\delta}{r}} = v_{av},$$

that is, the mean velocity tends to the value of the circular velocity, which as we know increases as the orbit sinks. Thus, the so-called "satellite paradox" can take place, namely, that during motion in the atmosphere the satellite experiences acceleration in the direction of its motion. The approximate nature of the velocity change for motion through the atmosphere is shown in Fig. 35, where n is the number of revolution.

The increase in the mean orbital velocity entails an increase in the kinetic energy $T = mv^2/2$. By writing out further the energy balance equation $h = T - U = T + V = mv^2/2 - \mu m/r$ and substituting this equation into the expression of the law of energy scattering, we get

$$\frac{dh}{dt} = \frac{d}{dt}(T + V) = \frac{d}{dt}\left(\frac{mv^2}{2} - \frac{\mu m}{r} \right).$$ /144

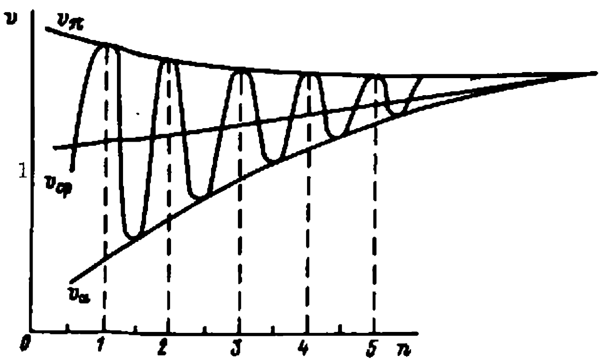

Fig. 35
Key: 1. v_{av}

Hence it follows that if there is a simultaneous increase in T due to an increase in the velocity v and a decrease in the total energy h resulting from the scattering of energy, the potential energy of the AES V = - U must fall off faster than the kinetic energy T increases.

It should be noted that the actual slowing down of AES in the atmosphere can be viewed as the result of applying a continuous series of infinitely small impulses, where the drag is directed along the tangent to this trajectory (tangential impulse) and is proportional to the instantaneous velocity. After applying each impulse, we get a new orbit with new constant $\tilde{h}$. The new ellipse is smaller than the former, and the tangent to it at the point of impulse application even has one focus coinciding with the focus of the initial ellipse.

The principles presented above can be illustrated by the following formulas. Suppose that at the perigee the velocity receives some increment Δv_π, so that $v_\pi' = v_\pi + \Delta v_\pi$. The sign of the increment is not known in advance. From the integral of energy when r_π = const, we have

$$h = v_\pi^2 - \frac{2\mu}{r_\pi} = -\frac{\mu}{a}, \quad \Delta h = 2 v_\pi \Delta v_\pi = \frac{\mu}{a^2} \Delta a,$$

$$a = \frac{r_\pi + r_\alpha}{2}, \quad \Delta a = \frac{\Delta r_\alpha}{2},$$

whence

$$\Delta r_\alpha = 2\Delta a = 2(2v_\pi \Delta v_\pi) a^2/\mu = 4 v_\pi \Delta v_\pi a^2/\mu. \tag{11.13}$$

This means that the increments Δr_α and Δr_π have the same signs, since the reduction in the velocity at the perigee necessarily entails a decrease in r_α. And this means the descent of the orbit. /145

Let us establish a relationship between the velocity increments at the perigee and at the apogee. Let us write out the

integral of energy for the apogee and the increment Δh for the variable r_α:

$$h = v_\alpha^2 - \frac{2\mu}{r_\alpha} = -\frac{\mu}{a},$$

$$\Delta h = 2 v_\alpha \Delta v_\alpha + \frac{2\mu}{r_\alpha^2} \Delta r_\alpha = \frac{\mu}{a^2} \Delta a = \frac{\mu}{a^2} \frac{\Delta r_\alpha}{2},$$

so that

$$2 v_\alpha \Delta v_\alpha = \mu\left(\frac{1}{2a^2} - \frac{2}{r_\alpha^2}\right)\Delta r_\alpha = \frac{\mu}{2a^2 r_\alpha^2}(r_\alpha^2 - 4a^2)\Delta r_\alpha =$$

$$= -\frac{\mu}{2a^2 r_\alpha^2}(r_\pi^2 + 2 r_\pi r_\alpha)\Delta r_\alpha.$$

Substituting in this expression in place of Δr_α, its value from Eq. (11.13), we find

$$\Delta v_\alpha = -\frac{v_\pi}{v_\alpha}\Delta v_\pi\left[\left(\frac{r_\pi}{r_\alpha}\right)^2 + 2\frac{r_\pi}{r_\alpha}\right]. \qquad (11.14)$$

From this formula it follows that the increments Δv_α and Δv_π have different signs. In addition, the orbit tends to a circular orbit, that is, $\frac{r_\pi}{r_\alpha} \to 1$ and $\frac{v_\pi}{v_\alpha} \to 1$. Therefore the impulse relation $\Delta v_\alpha = -3\Delta v_\pi$ obtains.

Problem 11.13. Determine the local circular and local parabolic velocities of a point with mass m over the surface of the Earth at the distance r_0 from its center if the motion occurs in the equatorial plane of the Earth. The Earth is taken as a compressed ellipsoid of revolution (spheroid).

Solution. The potential of a compressed Earth taken as an ellipsoid of revolution, to second-order terms relative to the geometrical compression of the Earth, is of the form /146

$$U = \frac{m\mu_\delta}{r}\left[1 + \frac{1}{3}J\left(\frac{R_\delta}{r}\right)^2(1 - 3\sin^2\varphi) + \frac{1}{10}D\left(\frac{R_\delta}{r}\right)^4\left(1 - 10\sin^2\varphi + \frac{35}{3}\sin^4\varphi\right)\right],$$

where $J = 1624 \cdot 10^{-6}$, and $D = 6 \cdot 10^{-6}$ are dimensionless coefficients containing the geometric compression α in the first and second powers; ϕ is the geometrical latitude of the points; and $R_{♁}$ is the equatorial radius of the Earth.

Let us write out the formula of the potential for $\phi = 0$ (equatorial plane):

$$U^0 = \frac{m\mu_♁}{r}\left[1 + \frac{1}{3}J\left(\frac{R_♁}{r}\right)^2 + \frac{1}{10}D\left(\frac{R_♁}{r}\right)^4\right]. \tag{11.15}$$

We know that the acceleration $\overline{w}$ of a point moving in a field of arbitrary potential force $\overline{F}$ in a system of spherical coordinates consists of three components:

$$\overline{w}_r = w_r\overline{r}^0 = \frac{F_r}{m}\overline{r}^0, \quad \overline{w}_\varphi = w_\varphi\overline{\varphi}^0 = \frac{F_\varphi}{m}\overline{\varphi}^0, \quad \overline{w}_\lambda = w_\lambda\overline{\lambda}^0 = \frac{F_\lambda}{m}\overline{\lambda}^0, \quad F_r = \frac{\partial U}{\partial r}, \quad F_\varphi = \frac{1}{r}\cdot\frac{\partial U}{\partial \varphi}, \quad F_\lambda = \frac{1}{r\cos\varphi}\cdot\frac{\partial U}{\partial \lambda}.$$

Suppose the point executes uniform circular motion in the equatorial plane of the Earth taken as an ellipsoid of revolution. The force of gravity F^0 ($\phi = 0$) in this case, as is true of the force of gravity of a spherical Earth, is a central force (see Fig. 36 a, b, and c), that is, $F^0_\phi = F^0_\lambda = 0$ and $F^0 = F^0_r$. Then the radial acceleration is simultaneously a normal acceleration, and the "longitudinal" component of acceleration corresponds to the tangential acceleration which is always zero for uniform circular motion. Thus,

$$w_r = -w_n = -\frac{v_{ci}^2}{r} = \frac{F^0}{m} = \frac{1}{m}\left(\frac{\partial U^0}{\partial r}\right) = \text{const} < 0, \quad w_\varphi = 0, \quad w_\lambda = w_\tau = \frac{dv_{ci}}{dt} = 0,$$

whence we can obtain the circular velocity

$$v_{ci} = \sqrt{-\frac{r_0}{m}\left(\frac{\partial U^0}{\partial r}\right)_{r=r_0}} = \sqrt{-\frac{r_0}{m}\left(F^0\right)_{r=r_0}} = \text{const} > 0.$$

The gravitational force $F^0 = F^0_r = \frac{\partial U^0}{\partial r}$ corresponding to it is found from Eq. (11.15): /147

$$F^0 = \frac{\partial U^0}{\partial r} = -\frac{m\mu_♁}{r^2}\left[1 + J\left(\frac{R_♁}{r}\right)^2 + \frac{1}{2}D\left(\frac{R_♁}{r}\right)^4\right], \tag{11.16}$$

so that the circular velocity is defined by the formula

$$v_{ci} = \sqrt{\frac{\mu_\delta}{r_0}\left[1 + J\left(\frac{R_\delta}{r_0}\right)^2 + \frac{1}{2}D\left(\frac{R_\delta}{r_0}\right)^4\right]} = \text{const} > 0. \qquad (11.17)$$

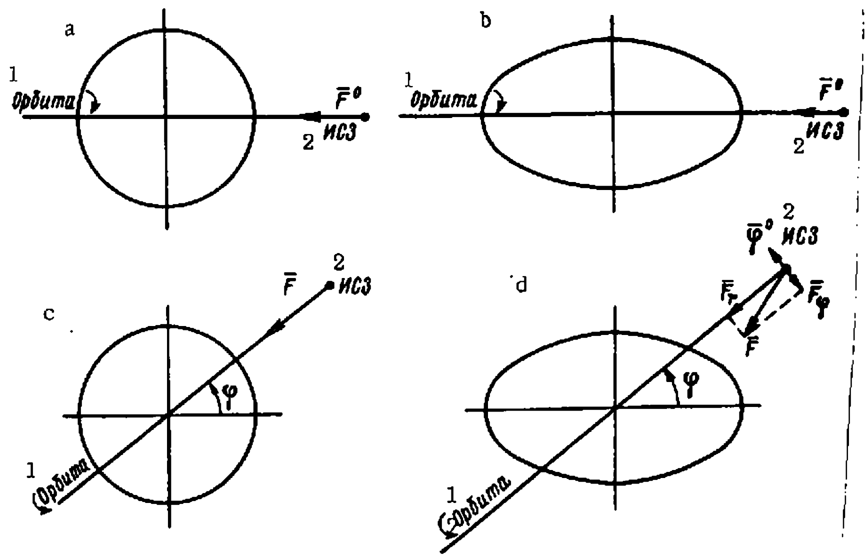

Fig. 36

Key: 1. Orbit
2. AES

To calculate the local parabolic velocity of a point, let us use the theorem of the change in the kinetic energy of a point as it moves "at infinity":

$$\frac{m v_\infty^2}{2} - \frac{m v_{par}^2}{2} = A_{r_0,\infty},$$

where $A_{r_0,\infty}$ is the work done by the gravitational force $\bar{F}^0$ in moving the point in the equatorial plane from a given surface of the potential level "at infinity", equal to $A_{r_0,\infty} = U^0(\infty) - U^0(r_0) < 0.$ /148

In this case $v_\infty = 0, U^0(\infty) = 0,$ so that $v_{par} = \sqrt{\frac{2}{m} U^0(r_0)}.$ Using Eq. (11.15) (where the distance $r = r_0$), let us determine the

parabolic velocity:

$$v_{par} = \sqrt{\frac{2\mu_\delta}{r_0}\left[1+\frac{1}{3}J\left(\frac{R_\delta}{r_0}\right)^2+\frac{1}{10}D\left(\frac{R_\delta}{r_0}\right)^4\right]}. \qquad (11.18)$$

A comparison of the formulas obtained for v_{ci} and v_{par} shows that the familiar formula $v_{par} = \sqrt{2}\, v_{ci}$ used by us in solving the problems in this collection corresponds only to the first term of a series in the expansion of the potential of the compressed Earth, that is, it is valid only for the spherical model of the Earth. Using Eqs. (11.17) and (11.18), let us write out the corrections to the values v_I = 7.90 km/sec and v_{II} = 11.19 km/sec for points on the equator ($r_0 = R_\delta$):

$$v_I = \sqrt{\frac{\mu_\delta}{R_\delta}\left(1+J+\frac{1}{2}D\right)} = 7{,}90\sqrt{1+1624\cdot 10^{-6}+3\cdot 10^{-6}} = 7{,}90\sqrt{1{,}001627},$$

$$v_{II} = \sqrt{\frac{2\mu_\delta}{R_\delta}\left(1+\frac{1}{3}J+\frac{1}{10}D\right)} = 11{,}19\sqrt{1+542\cdot 10^{-6}+0{,}3\cdot 10^{-6}} = 11{,}19\sqrt{1{,}000542}.$$

From these corrections it follows that the influence of the non-sphericity of the Earth for v_{II} is smaller than for v_I. As a point ascends above the Earth's surface this influence diminishes, which follows from the velocity formulas.

In passing from the equatorial plane ($\phi \neq 0$), the force of attraction of a compressed ellipsoid of revolution, owing to the disruption of symmetry, ceases to be a central force (Fig. 36 d) Here two components of the attractive [gravitational] force appear -- -- radial

$$F_r = \frac{\partial U}{\partial r} = -\frac{m\mu_\delta}{r^2}\left[1+J\left(\frac{R_\delta}{r}\right)^2(1-3\sin^2\varphi)+\frac{1}{2}D\left(\frac{R_\delta}{r}\right)^4\left(1-10\sin^2\varphi+\frac{35}{3}\sin^4\varphi\right)\right]$$

and "latitudinal"

$$F_\varphi = \frac{1}{r}\frac{\partial U}{\partial \varphi} = \frac{m\mu_\delta}{r^2}\left[-2J\left(\frac{R_\delta}{r}\right)^2\sin\varphi\cos\varphi+\frac{1}{30}D\left(\frac{R_\delta}{r}\right)^4(140\sin^3\varphi\cos\varphi-60\sin\varphi\cos\varphi)\right].$$ /149

The "longitudinal" component F_λ for the model of an ellipsoid of revolution is always equal to zero since $\overline{F}$ does not depend on λ.

Corresponding to these force components is the variable radial acceleration $w_r = \frac{F_r}{m} = \frac{1}{m}\left(\frac{\partial u}{\partial r}\right)_{r=r_0}$ and the nonzero acceleration $w_\varphi = \frac{F_\varphi}{m} = \frac{1}{mr_0}\left(\frac{\partial u}{\partial \varphi}\right)_{r=r_0,\,\varphi=\varphi_0}$, so that circular motion proves to be impossible for an ellipsoid of revolution when $\phi \neq 0$.

In this problem we have limited ourselves to examining the model of an ellipsoid of revolution, or a spheroid, taking the corresponding expansion to an accuracy of second-order terms. But if in the expansion (11.15) we take series terms that enable us not only to allow for the higher orders of compression for the model of a spheroid but also terms characterizing the triaxiality of the Earth (nonsphericity of the equator), the oblateness of the northern hemisphere, and so on, it becomes obvious that since in actuality the Earth's symmetry relative to the equatorial plane does not exist, neither does the strictly circular motion of an AES exist even in the equatorial plane. So for actual AES motions we can speak only of near-circular orbits.

Problem 11.14. A point is moving under the effect of the central attractive force $F = -m\left(\frac{\mu}{r^2} + \frac{\nu}{r^4}\right)$, where $\mu > 0$ and $\nu > 0$ are certain constants. Set up an equation of motion of the point using Binet's formula.

Solution. The physical significance of the problem posed can be established by using the formulas from problem 11.13 where Eq. (11.16) was derived for the force of attraction in the equatorial plane. From Eq. (11.16) it follows that this force, calculated to the first degree of the geometrical compression of the Earth, is of the form

$$F^0 = -\frac{m\mu_\delta}{r^2}\left[1 + J\left(\frac{R_\delta}{r}\right)^2\right], \quad J > 0.$$

This law of action of the force coincides with the law given in the conditions of the problem, if we take

$$\mu = \mu_\delta > 0, \quad \text{and} \quad \nu = \mu_\delta J R_\delta^2 > 0.$$

Thus, the first summand can be interpreted as the force of attraction of a spherical Earth, and the second summand can be interpreted as the incremental force of attraction of the equatorial access of the Earth's mass. /150

Using Binet's second formula (see Chapter Four), let us write

$$\frac{d^2u}{d\varphi^2} + u = -\frac{F}{mc^2u^2} = \left(\frac{\mu}{r^2} + \frac{\nu}{r^4}\right)\frac{1}{c^2u^2} = \frac{\mu}{c^2} + \frac{\nu u^2}{c^2},$$

where u = 1/r is Binet's variable, and $c = r^2\dot{\varphi}$ is the constant of areas. Hence

$$\frac{d^2u}{d\varphi^2} - \frac{\nu}{c^2}u^2 + u = \frac{\mu}{c^2} \tag{11.19}$$

(compare with Eqs. (11.4) from problem 11.6, and (4.11) from problem 4.10). This is a second-order nonlinear inhomogeneous equation of the type $y'' + ay^3 + by^2 + cy + d = 0\ (a = 0)$, which is insolvable in elementary functions. It can, for example, be reduced to an equation of the type $y'^2 + \frac{1}{2}ay^4 + \frac{2}{3}by^3 + cy^2 + 2dy + C = 0\ (a = 0)$, that is solvable in elliptical functions. However, from the physical point of view this equation is the equation of the perturbed motion of a point in a central field of attraction. In celestial mechanics, solutions of equations of perturbed motion are set up by using methods of successive approximations or methods of the variation of arbitrary constants. If these methods are applied to the solution of this equation (11.19), one must take as the first approximation the solution of the equation of unperturbed motion derived from (11.19) when ν = 0. This equation corresponds to the motion of a point in the field of attraction of a spherical Earth and its solution is an unperturbed Keplerian ellipse.

Problem 11.15. Determine the maximum possible period of revolution of an artificial Earth satellite T. What is the maximum duration of residence of this AES in the Earth's umbra?

Solution. We will assume the surface of the Earth's sphere of action to be the boundary of the domain of existence of the AES. Taking as the radius of the sphere ρ = 929,900 km (see Chapter Nine), which is 2.41 times greater than the distance of Earth to Moon $r_☾ = 384,400$ km, let us express the period of revolution of this AES in terms of the known period of revolution of the Moon around the Earth $T_☾ = 27.3$ days (see problem 10.7), using Kepler's third law:

$$T = T_☾\left(\frac{\rho}{r_☾}\right)^{3/2} = (2.41)^{3/2}T_☾ = 3.72 \cdot 27.3 = 101.6 \text{ days}.$$

Obviously, when $r > \rho$, the body will be converted into an artificial planet so that the period found can be assumed to be approximately of the possible periods. /151

Let us determine the duration of residence of an AES in the Earth's umbra. The length of the conical shadow of the Earth $OC = \varkappa$ (Fig. 37) can be found from the congruence of the triangles O'K'C' and OKC. By setting O'C ≈ K'C ≈ $\alpha + \varkappa$, we can write

$R_\odot / R_\oplus = (\alpha + \varkappa)/\varkappa$. Taking $R_\odot = 6.96 \cdot 10^5$ km, $R_\oplus = 6370$ km, and $\alpha = 149.6 \cdot 10^6$ km, we get $\frac{6.96 \cdot 10^5}{6.37 \cdot 10^3} = \frac{1.496 \cdot 10^8 + \varkappa}{\varkappa}$,

from whence $\varkappa \approx \frac{1.496 \cdot 10^8}{108.3} = 1.38 \cdot 10^6$ km.

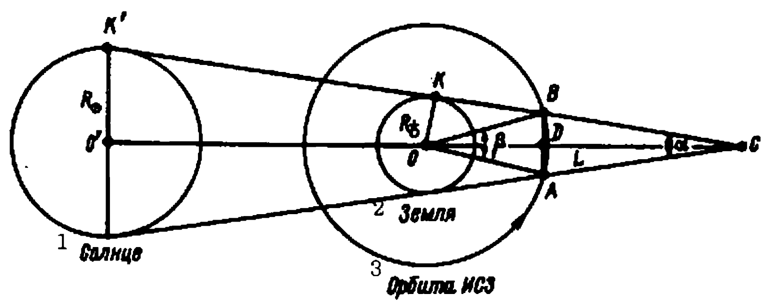

Fig. 37

Key: 1. Sun
2. Earth
3. Orbit of AES

The half-aperture angle of the cone of the Earth's umbra can be determined from ΔOKC:

$$\sin \frac{\alpha}{2} = \frac{R_\oplus}{\varkappa} = \frac{6.37 \cdot 10^3}{1.38 \cdot 10^6} = 0.0046, \quad \frac{\alpha}{2} = 0°16'.$$

We can readily see that if the length of the cone of the Earth's umbra is $\varkappa = 1.38 \cdot 10^6$ km, an object at a distance ρ = 930,000 km = $0.93 \cdot 10^6$ km from the Earth's center can fall within this cone, for a specific orbital inclination. The time spent in the umbra for an AES orbit situated in the plane of the ecliptic will be at a maximum when the orbital plane will lie in the plane of symmetry of the umbra cone (this orbital inclination is equal to the inclination of the ecliptic to the equator i = 23°.5).

The time of residence of this AES in the umbra can be determined as follows. Let us find the length of the segment AB (chords of circular segment) from the congruence of triangles OBD and OKC. It is

$$\frac{AB}{2R_8} = \frac{\varkappa - \rho}{\varkappa \cos\frac{\alpha}{2}},$$ /152

whence when $\cos\frac{\alpha}{2} \approx 1$ we have

$$AB = \frac{2R_8(\varkappa - \rho)}{\varkappa} = \frac{2 \cdot 6370\,(1.38 - 0.93)10^6}{1.38 \cdot 10^6} = 4.09 \cdot 10^3 \text{ km.}$$

We determine the corresponding central angle:

$$\sin\frac{\beta}{2} = \frac{AB}{2\rho} = \frac{4.09 \cdot 10^3}{2 \cdot 9.3 \cdot 10^5} = 0.0022,$$

$$\frac{\beta}{2} = 0°08', \quad \beta = 0°16' = 0.0047.$$

The proportion of the above-found period of revolution of the AES corresponding to its residence in the Earth's umbra is

$$\frac{\tau}{T} = \frac{\beta}{2\pi} = \frac{0.0047}{6.28} = 0.00075,$$

whence the time of residence in the umbra (maximum possible) is

$$\tau = 0.00075\,T = 0.00075 \cdot 101.6 = 0.076 \text{ day} = 1 \text{ hour } 50 \text{ min.}$$

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